

Moduli spaces of semistable sheaves of dimension 1 on \mathbb{P}^2 .

Yao YUAN

MPIM, Vivatsgasse 7, 53111 Bonn, GERMANY
yuayao@gmail.com

Abstract. Let $M(d, \chi)$ be the moduli space of semistable sheaves of rank 0, Euler characteristic χ and first Chern class dH ($d > 0$), with H the hyperplane class in \mathbb{P}^2 . We give a description of $M(d, \chi)$, viewing each sheaf as a class of matrices with entries in $\bigoplus_{i \geq 0} H^0(\mathcal{O}_{\mathbb{P}^2}(i))$. We show that there is a big open subset of $M(d, 1)$ isomorphic to a projective bundle over an open subset of a Hilbert scheme of points on \mathbb{P}^2 . Finally we compute the classes of $M(4, 1)$, $M(5, 1)$ and $M(5, 2)$ in the Grothendieck group of varieties, especially we conclude that $M(5, 1)$ and $M(5, 2)$ are of the same class.

1 Introduction.

Moduli spaces M of semistable sheaves of dimension 1 on surfaces are very interesting and many people have studied on them. On $K3$ or abelian surfaces, for a large number of M , Yoshioka has given explicitly the deformation classes of them in [7]. Le Potier studied a lot on M on \mathbb{P}^2 such as their Picard groups and rationalities in [4]. Drézet and Maican studied sheaves of dimension 1 on \mathbb{P}^2 with multiplicity 4, 5 and 6, via their locally free resolutions (see [1], [5] and [6]). But except few trivial cases, the classes of M on \mathbb{P}^2 in the Grothendieck group of varieties are not known.

Let $M(d, \chi)$ be the moduli space of semistable sheaves of rank 0, first Chern class dH ($d > 0$) and Euler characteristic χ on \mathbb{P}^2 . $M(d, \chi) \simeq M(d, \chi')$ if $\chi \equiv \chi' \pmod{d}$. There is a map $\pi : M(d, \chi) \rightarrow |dH|$ sending each sheaf to its support. Fibers of π over integral curves are isomorphic to their (compactified) Jacobians. But fibers of π over non-integral curves are not well understood.

In this paper we build a 1-1 correspondence between pure sheaves of dimension 1 on \mathbb{P}^2 and pairs (E, f) with E direct sums of line bundles on \mathbb{P}^2 and $f : E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow E$ injective, then after putting a stability condition on these pairs we can view $M(d, \chi)$ as the moduli space of semistable pairs (E, f) . From this point of view, we somehow avoid studying fibers of π over non-integral curves. However for general d , the moduli space is still so complicated to us that we are only able to describe a big open set of $M(d, \chi)$ with $\chi = 1$. We have the following proposition which is a generalization of Proposition 3.3.1 in [1] to all multiplicities.

Proposition 1.1 (Proposition 4.5). *There is an open subset $W^d \subset M(d, 1)$ with $M(d, 1) - W^d$ of codimension ≥ 2 , we have $W^d \simeq \mathbb{P}(\mathcal{V}^d)$, where \mathcal{V}^d is a vector bundle of rank $3d$ over $N_0^d := \text{Hilb}^{[\frac{(d-1)(d-2)}{2}]}(\mathbb{P}^2) - \Omega_{d-3}^{[\frac{(d-1)(d-2)}{2}]}$ with $\text{Hilb}^{[n]}(\mathbb{P}^2)$ the Hilbert scheme of n -points on \mathbb{P}^2 and $\Omega_k^{[n]}$ the closed subscheme of $\text{Hilb}^{[n]}(\mathbb{P}^2)$ parametrizing n -points lying on a curve of class kH .*

This proposition also implies an interesting corollary as follows.

Corollary 1.2 (Corollary 4.8). $\bar{d} := \frac{(d-1)(d-2)}{2}$. *Let C be an integral curve in \mathbb{P}^2 with degree d and I be an ideal sheaf of \bar{d} -points on C , then $h^0(\mathcal{H}om(I, \mathcal{O}_C)) = 1$ if and only if these \bar{d} -points do not lie on a curve of degree $d - 3$.*

Denote by $[X]$ the class of a variety X in the Grothendieck group of varieties. For $d \leq 5$ and $\text{g.c.d.}(d, \chi) = 1$, we compute $[M(d, \chi)]$ and get the following three theorems, with $\mathbb{L} := [\mathbb{A}^1]$ the class of the affine line.

Theorem 1.3 (Theorem 5.1). *For $d \leq 3$, $M(d, 1) = W^d$. Moreover $W^d \simeq |dH| \simeq \mathbb{P}^{3d-1}$ for $d = 1, 2$; $W^3 \simeq \mathcal{C}_3$ with \mathcal{C}_3 the universal curve in $\mathbb{P}^2 \times |3H|$.*

Theorem 1.4 (Theorem 5.2). $[M(4, 1)] = \sum_{i=0}^{17} b_{2i} \mathbb{L}^i$, such that

$$\begin{aligned} b_0 = b_{34} = 1; \quad b_2 = b_{32} = 2; \quad b_4 = b_{30} = 6; \\ b_6 = b_{28} = 10; \quad b_8 = b_{26} = 14; \quad b_{10} = b_{24} = 15; \\ b_{12} = b_{14} = b_{16} = b_{18} = b_{20} = b_{22} = 16. \end{aligned}$$

In particular the Euler number $e(M(4, 1))$ of the moduli space is 192.

Theorem 1.5 (Theorem 6.1). $[M(5, 1)] = [M(5, 2)] = \sum_{i=0}^{26} b_{2i} \mathbb{L}^i$, such that

$$\begin{aligned} b_0 = b_{52} = 1; \quad b_2 = b_{50} = 2; \quad b_4 = b_{48} = 6; \\ b_6 = b_{46} = 13; \quad b_8 = b_{44} = 26; \quad b_{10} = b_{42} = 45; \\ b_{12} = b_{40} = 68; \quad b_{14} = b_{38} = 87; \quad b_{16} = b_{36} = 100; \\ b_{18} = b_{34} = 107; \quad b_{20} = b_{32} = 111; \quad b_{22} = b_{30} = 112; \\ b_{24} = b_{26} = b_{28} = 113. \end{aligned}$$

In particular the Euler number of both moduli spaces is 1695.

Remark 1.6. *The Euler numbers $e(M(d, \chi))$ have been computed in [3] partially using physics arguments for $M(d, \chi)$ smooth. They have $e(M(d, \chi)) = (-1)^{\dim(M(d, \chi))} n_d^0$ with n_d^0 so-called BPS states of weight 0 for the local \mathbb{P}^2 (see Equation (4.2) and Table 4 in Section 8.3 in [3]). We see that our result accords with theirs for $d \leq 5$ and $\text{g.c.d.}(d, \chi) = 1$.*

The structure of the paper is arranged as follows. In Section 2, we construct a 1-1 correspondence between pure sheaves of dimension 1 and pairs (E, f) . The stability condition of (E, f) is given in Section 3. In Section 4 we study the big open set W^d in $M(d, 1)$ and prove Proposition 1.1 and Corollary 1.2. Theorem 1.3 and Theorem 1.4 are proved in Section 5 while Theorem 1.5 is proved in the last section—Section 6. We have Appendix A and B where we prove some technical lemmas used in Section 6.

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2 Pure sheaves of dimension 1 on \mathbb{P}^2 .

From now on except otherwise stated, a pair (E, f) on \mathbb{P}^2 always satisfies the following two conditions:

$$(1) E \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i) \text{ i.e. } E \text{ is a direct sum of line bundles on } \mathbb{P}^2; \quad (2.1)$$

$$(2) f \in \text{Hom}(E \otimes \mathcal{O}_{\mathbb{P}^2}(-1), E) \text{ and moreover } f \text{ is injective.} \quad (2.2)$$

Definition 2.1. *We say two pairs (E, f) and (E', f') are isomorphic if $E \simeq E'$ and there exists a following commutative diagram*

$$\begin{array}{ccc} E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f} & E \\ \varphi \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)} \downarrow & & \downarrow \phi \\ E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f'} & E' \end{array} \quad (2.3)$$

where φ and ϕ are two isomorphisms from E to E' .

Define two sets as follows

$$\mathcal{A} := \{\text{Isomorphism classes of pure sheaves of dimension 1}\};$$

$$\mathcal{B} := \{\text{Isomorphism classes of pairs } (E, f)\}.$$

It is obvious that we have a set-map θ from \mathcal{B} to \mathcal{A} sending each pair to its cokernel. We want to prove that θ is bijective. First we have the following lemma.

Lemma 2.2. *Let F be a sheaf of rank 0 and first Chern class dH ($d > 0$) on \mathbb{P}^2 , then F is pure of dimension 1 if and only if F lies in the following exact sequence with E_F a direct sum of line bundles.*

$$0 \rightarrow E_F \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow E_F \rightarrow F \rightarrow 0. \quad (2.4)$$

Proof. The “if” part is obvious: F in (2.4) is of rank 0 and has a locally free resolution of length 1, hence F is pure of dimension 1. To show the “only if”, it is enough to construct a sequence (2.4) for every pure sheaf F . We first follow the construction given by Le Potier in Proposition 3.10 in [4].

Since F is a torsion sheaf, we can take a point $x \in \mathbb{P}^2 - \text{Supp}(F)$ with $\text{Supp}(F)$ the support of F . Let $U := \mathbb{P}^2 - \{x\}$, then there is a projection $p : U \rightarrow \mathbb{P}^1$ and U is isomorphic to the total space of $\mathcal{O}_{\mathbb{P}^1}(1)$ on \mathbb{P}^1 . F is a sheaf of \mathcal{O}_U -modules. p_*F is pure on \mathbb{P}^1 because of the purity of F , and hence $p_*F \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(n_i)$. p_*F has a structure of $p_*\mathcal{O}_U$ -module which gives a morphism $f_1 : p_*F \rightarrow p_*F \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\widetilde{E}_F := p^*(p_*F)$. Pull f_1 back to U and define the following morphism

$$\widetilde{f} := (p^*f_1 - \lambda \text{id}_{\widetilde{E}_F}) \otimes p^*\text{id}_{\mathcal{O}_{\mathbb{P}^1}(-1)} : \widetilde{E}_F \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \widetilde{E}_F, \quad (2.5)$$

where λ is the canonical section of $p^*\mathcal{O}_{\mathbb{P}^1}(1)$. \widetilde{f} is injective and the cokernel is the sheaf F .

On the other hand, the complement of U in \mathbb{P}^2 is of codimension 2 and \widetilde{E}_F is direct sum of line bundles on U , hence both \widetilde{f} and \widetilde{E}_F can be extended to the whole \mathbb{P}^2 and we get a resolution of F on \mathbb{P}^2 as in (2.4) and $E_F \simeq j_*\widetilde{E}_F \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i)$ with $j : U \rightarrow \mathbb{P}^2$ the open immersion. Notice that $f|_x \simeq \mathbb{I}$ up to scalars, with f the extension of \widetilde{f} . Hence the lemma. \square

Lemma 2.2 implies that θ is surjective, then we have the injectivity by the following lemma.

Lemma 2.3. *Take any two exact sequences*

$$0 \longrightarrow E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f_1} E_1 \xrightarrow{g_1} F_1 \longrightarrow 0$$

$$0 \longrightarrow E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f_2} E_2 \xrightarrow{g_2} F_2 \longrightarrow 0,$$

with E_i direct sum of line bundles, then $F_1 \simeq F_2$ if and only if $(E_1, f_1) \simeq (E_2, f_2)$.

Proof. Again we only need to show the “only if” part. Firstly we see that $E_1 \simeq E_2$ as long as $F_1 \simeq F_2$, since E_i are direct sums of line bundles. We then want to construct the commutative diagram (2.3). The f_i ’s can be represented by square matrices with entries in $\bigoplus_{i \geq 0} H^0(\mathcal{O}_{\mathbb{P}^2}(i))$. After some invertible transformation, we can ask f_i to have the following form

$$f_i = \begin{pmatrix} \mathbb{I}_i & 0 \\ 0 & \mathbf{T}_i \end{pmatrix},$$

with \mathbb{I}_i the identity matrix and \mathbf{T}_i square matrix with entries in $\bigoplus_{i \geq 1} H^0(\mathcal{O}_{\mathbb{P}^2}(i))$. Hence we can write $E_i \simeq K_i \oplus M_i$ and $E_i \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \simeq K_i \oplus N_i$, such that f_i splits into the direct sum of an identity on K_i and a morphism $t_i : N_i \rightarrow M_i$ represented by \mathbf{T}_i . We then have the following exact sequence which is a minimal free resolution of F_i (see [2] Page 5 Definition)

$$0 \longrightarrow N_i \xrightarrow{t_i} M_i \xrightarrow{g_i|_{M_i}} F_i \longrightarrow 0.$$

Because of the uniqueness of the minimal free resolution (see [2] Page 6 Theorem 1.6), we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_1 & \xrightarrow{t_1} & M_1 & \xrightarrow{g_1|_{M_1}} & F_1 \longrightarrow 0 \\ & & \beta \downarrow \simeq & & \alpha \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & N_2 & \xrightarrow{t_2} & M_2 & \xrightarrow{g_2|_{M_2}} & F_2 \longrightarrow 0 \end{array} \quad (2.6)$$

Hence we have $K_1 \simeq K_2$ because $E_1 \simeq E_2$ and $M_1 \simeq M_2$. We define a map $\phi : E_1 \rightarrow E_2$ to be $\mathbb{I}_K \oplus \alpha$ with \mathbb{I}_K an isomorphism from K_1 to K_2 , and similarly we define the other map $\varphi \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)} : E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ to be $\mathbb{I}_K \oplus \beta$. Then we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f_1} & E_1 & \xrightarrow{g_1} & F_1 \longrightarrow 0 \\ & & \varphi \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)} \downarrow \simeq & & \phi \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & E_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f_2} & E_2 & \xrightarrow{g_2} & F_2 \longrightarrow 0 \end{array} \quad (2.7)$$

This finishes the proof of the lemma. \square

We finally get the following proposition.

Proposition 2.4. *There is a 1-1 correspondence between isomorphism classes of pure sheaves of dimension 1 and isomorphism classes of pairs (E, f) .*

3 The stability condition.

We put a stability condition on our pairs (E, f) , so that the map θ induces a bijection from semistable pairs to semistable sheaves. Given a pair (E, f) and its image F via θ , we write down the exact sequence

$$0 \longrightarrow E \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{f} E \xrightarrow{g} F \longrightarrow 0. \quad (3.1)$$

Recall that the slop of a torsion free sheaf E , $\mu(E)$, is defined as follows

$$\mu(E) := \frac{\deg(E)}{\text{rank}(E)};$$

and for a sheaf F of dimension 1 we have

$$\mu(F) := \frac{\chi(F)}{\deg(F)}.$$

We then have $\mu(E) + 1 = \mu(F)$ for E, F in sequence (3.1).

Definition 3.1. *We say a pair (E, f) is (semi)stable if for any $E' \subsetneq E$ and E' a direct sum of line bundles such that $f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, we have $\mu(E')(\leq) < \mu(E)$.*

Lemma 3.2. *θ induces a bijection from semistable pairs to semistable sheaves.*

Proof. As in sequence (3.1), we only need to prove that $\forall F' \subsetneq F$, $\exists E' \subset g^{-1}(F')$ a direct sum of line bundles and $f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, such that $F \simeq \text{coker}(f|_{E'})$ with $f|_{E'} : f^{-1}(E') \hookrightarrow E'$. Use the same notation as in the proof of Lemma 2.2, we see that p_*F' is a direct sum of line bundles on \mathbb{P}^1 and $p_*F' \subset p_*F$. Follow the construction in the proof of Lemma 2.2 and we see that $E' \simeq j_*p^*p_*F' \subset j_*p^*p_*F \simeq E$ and hence a direct sum of line bundles and moreover $f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, hence the lemma. \square

Let H be the hyperplane class in \mathbb{P}^2 and $u_{d,\chi}$ the class in the Grothendieck group of coherent sheaves on \mathbb{P}^2 with rank 0, first Chern class dH and Euler characteristic χ . Instead of sets, we consider the following two functors

$$\begin{aligned}\mathfrak{A} &:= \text{Functor of isomorphism classes of semistable sheaves of class } u_{d,\chi}; \\ \mathfrak{B} &:= \text{Functor of isomorphism classes of semistable pairs } (E, f) \text{ such that} \\ &\quad \text{rank}(E) = d \text{ and } c_1(E) = (\chi - d)H.\end{aligned}$$

We see that map θ induces a functor transformation from \mathfrak{B} to \mathfrak{A} which is an isomorphism of functors.

Proposition 3.3. $\mathfrak{B} \simeq \mathfrak{A}$.

From now on denote by $M(d, \chi)$ the moduli space parametrizing semistable sheaves of class $u_{d,\chi}$, we can also view it as the moduli space parametrizing semistable pairs (E, f) such that $\text{rank}(E) = d$ and $c_1(E) = (\chi - d)H$.

We write down the following lemma for future use.

Lemma 3.4. *Let (E, f) be a semistable pair, then for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') \leq \mu(E)$. In particular write $E \simeq \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ with $a_i > 0$ for all i and $n_1 > n_2 > \dots > n_k$, then $n_i - n_{i+1} = 1$ for all $1 \leq i \leq k-1$.*

Proof. Since $D' \simeq D''$ and they are both direct summands of E , $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'' \Leftrightarrow f^{-1}(D'') = D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \simeq D'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$. Hence the first statement.

If $\exists i_0$ such that $n_{i_0} - n_{i_0+1} \geq 2$, then $f((\bigoplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \bigoplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ and $\mu(\bigoplus_{i=1}^{i_0} \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}) > \mu(E)$, which is a contradiction. Hence the lemma. \square

4 A big open subset in $M(d, 1)$.

The moduli space $M(d, \chi)$ is irreducible (see [4] Theorem 3.1) and the stable locus $M(d, \chi)^s$ is smooth. $M(d, \chi) \simeq M(d, \chi')$ if $\chi \equiv \pm \chi' \pmod{d}$. $M(d, \chi) = M(d, \chi)^s$ if and only if $\text{g.c.d.}(d, \chi) = 1$. Hence $M(d, 1) = M(d, 1)^s$ and the moduli space is smooth of dimension $d^2 + 1$ for all $d \geq 1$. Moreover there is a universal sheaf on $M(d, 1) \times \mathbb{P}^2$ by Theorem 3.19 in [4].

We want to give a concrete description of an open subset in $M(d, 1)$ where the pair (E, f) satisfies that $E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$. We first rephrase the stability condition for $E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$ as follows.

Lemma 4.1. *If $E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$, then (E, f) is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. Because of Lemma 3.4, we only need to prove the “if”. It is enough to show that for any $E' \subset E$ such that E' is a direct sum of line bundles and $\mu(E') > \mu(E)$, E' is a direct summand of E . But we get this by direct observation. Hence the lemma. \square

Define $\widetilde{W}^d := \{(E, f) | E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}\} \subset M(d, 1)$, we then have the following lemma.

Lemma 4.2. *$M(d, 1) - \widetilde{W}^d$ is of codimension ≥ 2 in $M(d, 1)$.*

Proof. For any point $x \in \mathbb{P}^2$, denote Y_x to be the open subset of $M(d, 1)$ where the pair (E, f) satisfies that $x \notin \text{Supp}(\text{coker}(f))$. Because $M(d, 1)$ can be covered by finitely many Y_x , it is enough to show that $Y_x \cap (M(d, 1) - \widetilde{W}^d)$ is of codimension ≥ 2 in Y_x . But this has been proved in [4] as Proposition 3.14. Hence the lemma. \square

Now we look at a pair (E, f) in \widetilde{W}^d . By Lemma 4.1 $f(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \not\subset \mathcal{O}_{\mathbb{P}^2}$, hence the induced map $f_{\text{restr}} : \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$ is nonzero, hence we can ask f to identify $\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with a summand $\mathcal{O}_{\mathbb{P}^2}(-1)$ in E , hence f can be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ A & 0 & B \end{pmatrix},$$

where A is a $(d-1) \times 1$ matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and B a $(d-1) \times (d-2)$ matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. B provides a morphism $f_B : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus d-2}$. The stability condition in Lemma 4.1 is equivalent to the following condition

Condition 4.3. $f_B(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d'-1}) \not\subseteq \mathcal{O}_{\mathbb{P}^2}^{\oplus d'-2}$ for any $0 \neq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d'-1} \subsetneq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$.

We consider the morphism $f_{B^t} : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1}$ represented by the transform of B, B^t . Then Condition 4.3 is equivalent to the following condition

Condition 4.4. $f_{B^t}^{-1}(\mathcal{O}_{\mathbb{P}^2}^{\oplus d'-1}) \not\subseteq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d'-1}$ for any $0 \neq \mathcal{O}_{\mathbb{P}^2}^{\oplus d'-1} \subsetneq \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1}$.

We have the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} & \xrightarrow{f_{B^t}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1} & \xrightarrow{f_q} & Q_f \longrightarrow 0 \\
& & & & \uparrow f_{A^t} & \nearrow \sigma_f := f_q \circ f_{A^t} & \\
& & & & \mathcal{O}_{\mathbb{P}^2}(-2) & &
\end{array} \quad (4.1)$$

The injectivity of f_{B^t} is because of the injectivity of f . Denote $F := \text{coker}(f)$ and $F^\vee := \mathcal{E}xt^1(F, \mathcal{O}_{\mathbb{P}^2})$, then we have $F^{\vee\vee} \simeq F$ and moreover F and F^\vee are determined by each other (see [8] Lemma A.0.13). We write down a bigger commutative diagram as follows

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-2) & \longrightarrow 0 \\
& \downarrow \simeq & & \downarrow f_{B^t} \oplus f_{A^t} & & \downarrow \sigma_f & \\
0 \rightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} & \xrightarrow{f_{B^t}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1} & \xrightarrow{f_q} & Q_f & \longrightarrow 0 \\
& & & \downarrow & & \downarrow \delta & \\
& & & F^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{\simeq} & F^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-2) & \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array} \quad (4.2)$$

We see that the isomorphism classes of F^\vee are parametrized by pairs (Q_f, σ_f) , hence so are the isomorphism classes of F .

Define $W^d := \{[(E, f)] \in \widetilde{W}^d \mid Q_f \text{ is torsion free}\}$. Then we have the following proposition.

Proposition 4.5. $W^d \simeq \mathbb{P}(\mathcal{V}^d)$, where \mathcal{V}^d is a vector bundle of rank $3d$ over $N_0^d := \text{Hilb}^{[\frac{(d-1)(d-2)}{2}]}(\mathbb{P}^2) - \Omega_{d-3}^{[\frac{(d-1)(d-2)}{2}]}$ with $\text{Hilb}^{[n]}(\mathbb{P}^2)$ the Hilbert scheme of n -points on \mathbb{P}^2 and $\Omega_k^{[n]}$ the closed subscheme of $\text{Hilb}^{[n]}(\mathbb{P}^2)$ parametrizing n -points lying on a curve of class kH .

Proof. We need to parametrize all the pairs (Q_f, σ_f) in diagram (4.1) with Q_f torsion free.

Define $\bar{d} := \frac{(d-1)(d-2)}{2}$. If Q_f in diagram (4.1) is torsion free, then by direct calculation we know that $Q_f \simeq I_{\bar{d}} \otimes \mathcal{O}_{\mathbb{P}^2}(d-2)$, with I_n the ideal sheaf of a 0-dimensional subscheme of length n on \mathbb{P}^2 . Condition 4.4 is satisfied automatically for Q_f torsion free.

We first want to show that N_0^d parametrizes all the torsion free Q_f in diagram (4.1), which follows from the following lemma.

Lemma 4.6. $I_{\bar{d}}(d-2) := I_{\bar{d}} \otimes \mathcal{O}_{\mathbb{P}^2}(d-2)$ has the following resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus d-1} \rightarrow I_{\bar{d}}(d-2) \rightarrow 0. \quad (4.3)$$

if and only if $H^0(I_{\bar{d}}(d-3)) = 0$.

Proof. We only need to show the “if” because once $I_{\bar{d}}(d-2)$ lies in sequence (4.3), $H^0(I_{\bar{d}}(d-3)) \simeq H^1(\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus d-2}) = 0$.

We have the following exact sequence

$$0 \rightarrow I_{\bar{d}}(i) \rightarrow \mathcal{O}_{\mathbb{P}^2}(i) \rightarrow \mathcal{O}_Z \rightarrow 0, \forall i \in \mathbb{Z}, \quad (4.4)$$

with Z a 0-dimensional subscheme of length \bar{d} . Take global sections on (4.4) and we get

$$0 \rightarrow H^0(I_{\bar{d}}(i)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(i)) \rightarrow H^0(\mathcal{O}_Z), \forall i \in \mathbb{Z} \quad (4.5)$$

By direct calculation we get $h^0(I_{\bar{d}}(d-2)) \geq d-1$, hence these \bar{d} points always lie on a curve of degree $d-2$. By Corollary 3.9 in [2] Page 38, we know that $I_{\bar{d}}$ can be generated by $d-1$ elements, hence $H^0(I_{\bar{d}}(d-3)) = 0 \Rightarrow h^0(I_{\bar{d}}(d-2)) = d-1$ and any $d-1$ linearly independent elements in $H^0(I_{\bar{d}}(d-2))$ generate $I_{\bar{d}}$ as a graded S -module, where S denotes the graded ring $\mathbb{C}[x_0, x_1, x_2]$. Moreover these $d-1$ generators form a minimal set of generators.

As a graded S -module, $I_{\bar{d}}$ has a free resolution of length 1 by Proposition 3.1 in [2] Page 32. Moreover we know that $I_{\bar{d}}$ is generated by $d-1$ elements in degree $d-2$, we have the following exact sequence of graded S -modules

$$0 \rightarrow \text{Ker} \rightarrow S(2-d)^{\oplus d-1} \rightarrow I_{\bar{d}} \rightarrow 0. \quad (4.6)$$

where Ker is a free graded S -module of rank $d-2$. By direct calculation we get that $\text{Ker} \simeq S(1-d)^{\oplus d-2}$. Hence we see that $I_{\bar{d}}(d-2)$ lies in sequence (4.3). This finishes the proof of the lemma. \square

One can see that σ_f up to scalars can be viewed as an element in $\mathbb{P}H^0(Q_f(2))$ which is exactly $\det(f)$ up to scalars.

Because N_0^d is open in $\text{Hilb}^{[\bar{d}]}(\mathbb{P}^2)$, on $\mathbb{P}^2 \times N_0^d$ we have a universal sheaf $\mathcal{I}_{\bar{d}}$ which restricted to the fiber over each point $[I_{\bar{d}}] \in N_0^d$ is the ideal sheaf $I_{\bar{d}}$. We have the diagram

$$\begin{array}{ccc} \mathcal{I}_{\bar{d}} & \longrightarrow & \mathbb{P}^2 \times N_0^d \\ & \searrow q & \downarrow p \\ \mathbb{P}^2 & & N_0^d \end{array} \quad (4.7)$$

We see that $R^i p_*(\mathcal{I}_{\bar{d}} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d)) = 0$ for $i \geq 1$ and $p_*(\mathcal{I}_{\bar{d}} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d))$ is locally free of rank $3d$. Define $\mathcal{V}^d := p_*(\mathcal{I}_{\bar{d}} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d))$. There is a 1-1 correspondence between points in $\mathbb{P}(\mathcal{V}^d)$ and isomorphism classes of (Q_f, σ_f) with Q_f torsion free. To prove the proposition, it is enough to construct a family \mathcal{F} of stable sheaves of class $u_{d,1}$ over $\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d)$.

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d) & \xrightarrow{id_{\mathbb{P}^2} \times \pi} & \mathbb{P}^2 \times N_0^d \\ \tilde{p} \downarrow & & \downarrow p \\ \mathbb{P}(\mathcal{V}^d) & \xrightarrow{\pi} & N_0^d \end{array} \quad \begin{array}{c} \searrow q \\ \mathbb{P}^2 \end{array} \quad (4.8)$$

Denote $\mathcal{O}_\pi(1)$ to be the relative polarization on $\mathbb{P}(\mathcal{V}^d)$ over N_0^d . We have a natural exact sequence on $\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d)$ as follows

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}^d)} \rightarrow \tilde{p}^* \mathcal{O}_\pi(1) \otimes (id_{\mathbb{P}^2} \times \pi)^*(\mathcal{I}_{\bar{d}} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(d)) \rightarrow \mathcal{F}^\vee \rightarrow 0. \quad (4.9)$$

We see that fiberwise (4.9) is the first vertical exact sequence from the right hand side in (4.2) tensored by $\mathcal{O}_{\mathbb{P}^2}(2)$. Hence \mathcal{F}^\vee is a family of stable sheaves of class $u_{d,1}^\vee$. We get \mathcal{F} by taking the dual. Hence the proposition. \square

We now have a concrete description of the open subset W^d of the moduli space $M(d, 1)$.

Proposition 4.7. *$M(d, 1) - W^d$ is of codimension ≥ 2 in $M(d, 1)$.*

Proof. By Lemma 4.2, we only need to show that $\widetilde{W}^d - W^d$ is of codimension at least 2 in $M(d, 1)$. Let $|dH|$ denote the linear system of divisors of class dH , then non-integral curves form a closed subset of codimension ≥ 2 in $|dH|$. Therefore by Proposition 2.8 and Lemma 3.2 in [4], we know that stable sheaves with non-integral supports form a closed subset of codimension ≥ 2 in $M(d, 1)$. Hence it is enough to show that if Q_f in (4.1) is not torsion free, then $Supp(F) = Supp(coker(f))$ is non-integral.

Denote by T_f the torsion of Q_f . Since Q_f has a free resolution of length 1, T_f must be a pure sheaf supported on a curve in $|d'H|$. Look back to diagram (4.2), the map δ restricted to T_f gives a nonzero element in $\text{Hom}(T_f, F^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-2))$. If $d' < d$, then $Supp(F^\vee) = Supp(F)$ can not be integral. Now we look at the following exact sequence

$$0 \rightarrow T_f \rightarrow Q_f \rightarrow Q_f^{tf} \rightarrow 0.$$

The torsion free sheaf Q_f^{tf} has the form $I_n(m) := I_n \otimes \mathcal{O}_{\mathbb{P}^2}(m)$ such that $m + d' = \deg(Q_f) = d - 2$. On the other hand, the surjective morphism f_q induces a surjective morphism from $\mathcal{O}_{\mathbb{P}^2}^{\oplus d-1}$ to Q_f^{tf} , hence $m \geq 1$ and thus $d' < d - 2 < d$. Therefore $\text{Supp}(F)$ can not be integral and this finishes the proof. \square

If $Q_f \simeq I_{\bar{d}}(d-2)$, we can see that $\text{coker}(f) = F \simeq \mathcal{E}xt^1(I_{\bar{d}}(d)/I_C(d), \mathcal{O}_{\mathbb{P}^2})$ with I_C the ideal sheaf of the supporting curve C of F . Hence by coherent duality $F \simeq \mathcal{H}om(I_{\bar{d}|C}, \mathcal{O}_C)$ with $I_{\bar{d}|C}$ the ideal sheaf of \bar{d} -points on C . Therefore we have the following interesting corollary.

Corollary 4.8. *Let C be an integral curve in \mathbb{P}^2 with degree d and I be an l sheaf of \bar{d} -points on C , then $h^0(\mathcal{H}om(I, \mathcal{O}_C)) = 1$ if and only if these \bar{d} -points do not lie on a curve of degree $d - 3$.*

Proof. For C integral, $\mathcal{H}om(I, \mathcal{O}_C)$ is torsion free of rank 1 hence stable and $\chi(\mathcal{H}om(I, \mathcal{O}_C)) = 1$, therefore $[\mathcal{H}om(I, \mathcal{O}_C)] \in M(d, 1)$. On the other hand $\forall [F] \in M(1, d)$, $h^0(F) = 1 \Leftrightarrow [F] \in \widetilde{W}^d$. Since C is integral, $h^0(\mathcal{H}om(I, \mathcal{O}_C)) = 1 \Leftrightarrow [\mathcal{H}om(I, \mathcal{O}_C)] \in W^d \Leftrightarrow H^0(I(d-3)) = 0$. This finishes the proof. \square

5 $M(d, 1)$ with $d \leq 4$.

In this section we study $M(d, 1)$ with $d \leq 4$, viewing it as the moduli space of stable pairs (E, f) . Notice that for $d \leq 4$, up to isomorphism $M(d, 1)$ is the only moduli space such that there is no strictly semistable locus, since $M(d, \chi) \simeq M(d, \chi')$ if $\chi \equiv \pm \chi' \pmod{d}$.

For $d \leq 3$, $M(d, 1)$ is very easy to understand and the following theorem is already known by Theorem 3.5 and Theorem 5.1 in [4]. But however using our new description we give another proof. Recall that we have defined a big open subset $W^d \subset M(d, 1)$ in the previous section.

Theorem 5.1. *For $d \leq 3$, $M(d, 1) = W^d$. Moreover $W^d \simeq |dH| \simeq \mathbb{P}^{3d-1}$ for $d = 1, 2$; $W^3 \simeq \mathcal{C}_3$ with \mathcal{C}_3 the universal curve in $\mathbb{P}^2 \times |3H|$.*

Proof. By Lemma 3.4 we see that for $d \leq 3$, the sheaf E in a stable pair (E, f) can only have the form $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus d-1}$. From the proof of Proposition 4.7 we see that the torsion of Q_f can only be supported on a curve of degree no bigger than $d - 3$, hence Q_f is always torsion free for $d \leq 3$. Hence the first statement. By direct observation, we get the form of W^d for $d \leq 3$. This finishes the proof. \square

Denote by $[X]$ the class of a variety X in the Grothendieck group of varieties. Define $\mathbb{L} := [\mathbb{A}^1]$ with \mathbb{A}^1 the affine line. We have the following theorem.

Theorem 5.2. $[M(4, 1)] = \sum_{i=0}^{17} b_{2i} \mathbb{L}^i$, such that

$$\begin{aligned} b_0 &= b_{34} = 1; & b_2 &= b_{32} = 2; & b_4 &= b_{30} = 6; \\ b_6 &= b_{28} = 10; & b_8 &= b_{26} = 14; & b_{10} &= b_{24} = 15; \\ b_{12} &= b_{14} = b_{16} = b_{18} = b_{20} = b_{22} = 16. \end{aligned}$$

In particular the Euler number $e(M(4, 1))$ of the moduli space is 192.

Remark 5.3. Theorem 5.2 only gives the class of $M(4, 1)$ in the Grothendieck group of varieties. But we do not know whether the moduli space has a cell decomposition.

To prove Theorem 5.2, we view $M(4, 1)$ as the moduli space of stable pairs (E, f) and stratify $M(4, 1)$ by the form of E . According to Lemma 3.4, there are two strata defined as follows.

$$\begin{aligned} M_1 &:= \{[(E, f)] \in M(4, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}\}; \\ M_2 &:= \{[(E, f)] \in M(4, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}. \end{aligned}$$

Lemma 5.4. A pair (E, f) with $\text{rank}(E) = 4$ and $\deg(E) = -3$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.

Proof. We only need to prove the lemma for $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. We want to show that if $\exists E' \subset E$, E' is a direct sum of line bundles with $\mu(E') > \mu(E)$ and $f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then $\exists D, D' \subset E$ as direct summands with $D \simeq D'$ and $\mu(D) > \mu(E)$, such that $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'$. With no loss of generality, we assume that E' has the form $\bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ with $a_i > 0$ and $n_i - n_{i+1} = 1$.

Let $E' \simeq E'' \subset E$ and E'' is not a direct summand of E . Then E'' has to be one of the following two cases:

- (1) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;
- (2) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

Let E'' be in case (1). If E' is a direct summand of E , then $f^{-1}(E') \neq E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ because by Nakayama's lemma $E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \subset f^{-1}(E') \Rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \subset f^{-1}(E')$.

If $E' = E''$ and $f^{-1}(E'') = E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then again by Nakayama's lemma we have $f(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$. Hence we get $D = D' = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$.

If $E' = E''$ and $f^{-1}(E')$ is a direct summand of $E \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ isomorphic to $E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then we have $D = f^{-1}(E') \otimes \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} = D'$ and $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'$, since there is no nonzero morphism from $\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ to $\mathcal{O}_{\mathbb{P}^2}(-2)$. Hence case (1) is done.

Case (2) is analogous. Hence the lemma. \square

For a pair $(E, f) \in M_2$, similarly as we did in the previous section, we ask f to be represented by the following matrix

$$\begin{pmatrix} b_1 & b_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \end{pmatrix}, \quad (5.1)$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. The injectivity of f implies that $\det(f) = b_1 a_2 - b_2 a_1 \neq 0$. Moreover by Lemma 5.4 we can see that (E, f) is stable if and only if $kb_1 \neq k'b_2$ for any $(k, k') \in \mathbb{C}^2 - \{0\}$.

Lemma 5.5. $[M_2] = [\mathbb{P}^2 \times \mathbb{P}^{13}]$.

Proof. We have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{(b_1, b_2)} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} & \xrightarrow{f_r} & R_f \longrightarrow 0 \\ & & & & \uparrow (a_1, a_2) & \nearrow \omega_f := f_r \circ (a_1, a_2) & \\ & & & & \mathcal{O}_{\mathbb{P}^2}(-3) & & \end{array} \quad (5.2)$$

Since $kb_1 \neq k'b_2$ for any $(k, k') \in \mathbb{C}^2 - \{0\}$, we have $R_f \simeq I_1(1)$. On the other hand, every $I_1(1)$ can be put in sequence (5.2). Hence $\text{Hilb}^{[1]}(\mathbb{P}^2)$ parametrizes the isomorphism classes of R_f . By analogous argument to the the proof of Proposition 4.5, we see that isomorphism classes of (R_f, ω_f) are parametrized by a projective bundle over $\text{Hilb}^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_1(4))) \simeq \mathbb{P}^{13}$. Hence the lemma follows as a direct consequence. \square

The big open subset W^4 as defined in the previous section is contained in M_1 . We have $[W^4] = [(\text{Hilb}^{[3]}(\mathbb{P}^2) - \Omega_1^{[3]}) \times \mathbb{P}^{11}]$ by Proposition 4.5.

Lemma 5.6. $\Omega_1^{[3]} \simeq \mathcal{C}_1^{[3]}$ with $\mathcal{C}_1^{[3]}$ the relative Hilbert scheme of 3-points on the universal family $\mathcal{C}_1 \subset \mathbb{P}^2 \times |H|$, and hence $[\Omega_1^{[3]}] = [\mathbb{P}^2 \times \mathbb{P}^3]$.

Proof. We have a natural map $\xi : \mathcal{C}_1^{[3]} \rightarrow \Omega_1^{[3]}$. ξ is an isomorphism because there is at most one curve in $|H|$ passing through any 3 points. $\mathcal{C}_1 \rightarrow |H|$ is a \mathbb{P}^1 -bundle, hence the map $p : \mathcal{C}_1^{[3]} \rightarrow |H|$ is a projective bundle with fibers isomorphic to $(\mathbb{P}^1)^{[3]} \simeq \mathbb{P}^3$, therefore $[\Omega_1^{[3]}] = [\mathcal{C}_1^{[3]}] = [\mathbb{P}^2 \times \mathbb{P}^3]$. \square

Now we only need to compute $[M_1 - W^4]$. Look back to diagram (4.1), we see that we can parametrize pairs (E, f) in M_1 by the other pairs (Q_f, σ_f) . We want to see what Q_f will be if it is not torsion free for $d = 4$. From the proof of Proposition 4.7 we see that the torsion of Q_f can only be supported on a curve of degree no bigger than $d - 3 = 1$. We write down the following exact sequence

$$0 \rightarrow T_f \rightarrow Q_f \rightarrow Q_f^{tf} \rightarrow 0,$$

with T_f the torsion of Q_f and Q_f^{tf} a torsion free sheaf of rank 1.

Since T_f is supported on a curve in $|H|$ and $h^0(T_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \leq h^0(Q_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$, $T_f \simeq \mathcal{O}_H(t) \simeq \mathcal{O}_{\mathbb{P}^1}(t)$ with $t \leq 0$. Let $Q_f^{tf} \simeq I_n(m)$ with $m > 0, n \geq 0$. Then we have $m = 1$ and $n - t = 1$ by direct calculation.

If $t = 0, n = 1$, then we have the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & \mathcal{O}_H \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \longrightarrow & Q_f \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} & \longrightarrow & I_1(1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \quad (5.3)$$

which contradicts Condition 4.4. Hence we have $t = -1, n = 0$ and Q_f lies in the following exact sequence

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow Q_f \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0. \quad (5.4)$$

$\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H(-1)) \simeq H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{C}$, so for a fixed projective line \mathbb{P}^1 in $|H|$, if sequence (5.4) does not split, Q_f is unique up to isomorphism.

Lemma 5.7. *Q_f in (5.4) also lies in the following exact sequence (5.5) if and only if sequence (5.4) does not split.*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \rightarrow Q_f \rightarrow 0. \quad (5.5)$$

Proof. We see that if $Q_f \simeq \mathcal{O}_H(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, it certainly can not lie in sequence (5.5). If sequence (5.4) does not split, then Q_f is unique, so we only need to construct the sequence in (5.5) with Q_f contains $\mathcal{O}_H(-1)$ as its torsion. Write $\mathbb{P}^2 = \mathbf{Proj} \mathbb{C}[x_0, x_1, x_2]$. With no loss of generality we assume that $\mathcal{O}_H(-1)$ is supported on $\{x_0 = 0\}$, then the following matrix represents a morphism $f_{B^t} : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ such that $\text{coker}(f_{B^t})$ contains $\mathcal{O}_{\{x_0=0\}}(-1)$ as its torsion.

$$f_{B^t} := \begin{pmatrix} x_0 & 0 \\ x_1 & x_2 \\ 0 & x_0 \end{pmatrix}. \quad (5.6)$$

This finishes the proof. \square

Remark 5.8. f_{B^t} defined in (5.6) also satisfies the stability condition i.e. Condition 4.4.

Lemma 5.9. Decompose $|H|$ into cells and write $|H| = \cup_{i=0}^2 \mathbb{A}^i$. Then \mathbb{A}^i parametrizes isomorphism classes of Q_f such that pairs $(Q_f, \sigma_f) \in M_1 - W^4$ and T_f are supported on curves in $\mathbb{A}^i \subset |H|$.

Proof. Lemma 5.7 implies that there is a 1-1 correspondence between isomorphism classes of Q_f and points in $|H|$. We need to decompose $|H|$ into cells so that we have a universal family over $\mathbb{P}^2 \times \mathbb{A}^i$ for each i .

We have the following diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \longrightarrow & \mathbb{P}^2 \times |H|, \\ & \searrow q & \downarrow p \\ \mathbb{P}^2 & & |H| \end{array} \quad (5.7)$$

with \mathcal{C}_1 the universal curve of degree 1.

$\text{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H(-1)) = 0$ for all $i \neq 1$ and $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_H(-1)) \simeq \mathbb{C}$, therefore $L := \mathcal{E}xt_p^1(q^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1))$ is a line bundle on $|H|$. Moreover $ch(L) = -ch(R^\bullet p_* \circ R^\bullet \mathcal{H}om(q^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)))$, so by Grothendieck-Hirzbruch-Riemann-Roch Theorem we can compute and get that $c_1(L) = c_1(\mathcal{O}_{|H|}(1))$ and hence $L \simeq \mathcal{O}_{|H|}(1)$.

L has a nowhere vanishing global section on each \mathbb{A}^i , in other words, we have an exact sequence on $\mathbb{P}^2 \times \mathbb{A}^i$

$$0 \rightarrow \mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-1)|_{\mathbb{P}^2 \times \mathbb{A}^i} \rightarrow \mathcal{Q}^i \rightarrow q^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^2 \times \mathbb{A}^i} \rightarrow 0, \quad (5.8)$$

such that restricted on the fiber over any point $y \in \mathbb{A}^i$ it does not split. Hence \mathcal{Q}^i is the family we want and hence the lemma. \square

We rewrite diagram (4.1) for $d = 4$ as the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \xrightarrow{f_{Bt}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \xrightarrow{f_q} & Q_f \longrightarrow 0 \\
& & & & \uparrow f_{At} & \nearrow \sigma_f = f_q \circ f_{At} & \\
& & & & \mathcal{O}_{\mathbb{P}^2}(-2) & &
\end{array} \tag{5.9}$$

Denote $\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$ to be the union of the projective bundles $\mathbb{P}(p_*(\mathcal{Q}^i \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)|_{\mathbb{P}^2 \times \mathbb{A}^i}))$ over \mathbb{A}^i with fibers isomorphic to $\mathbb{P}H^0(Q_f(2)) \simeq \mathbb{P}^{11}$. Analogously, isomorphism classes of pairs (Q_f, σ_f) can be parametrized by $\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$. However $\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2))) \not\subset M(4, 1)$. Look back to diagram (4.2), it is easy to see that

$$\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2))) \cap M(4, 1) = \{[(Q_f, \sigma_f)] | \text{Im}(\sigma_f) \not\subset T_f\}, \tag{5.10}$$

with $\text{Im}(\sigma_f)$ the image of σ_f and T_f the torsion of Q_f .

The complement of (5.10) in $\mathbb{P}(p_*(\mathcal{Q} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$ is the union of the projective bundles $\mathbb{P}(p_*(\mathcal{O}_{C_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^2 \times \mathbb{A}^i}))$ over \mathbb{A}^i with fibers isomorphic to $\mathbb{P}H^0(\mathcal{O}_H(1)) \simeq \mathbb{P}^1$. We then have the following lemma as a straightforward consequence.

Lemma 5.10. $[M_1 - W^4] = [\mathbb{P}^2 \times \mathbb{P}^{11} - \mathbb{P}^2 \times \mathbb{P}^1]$.

Proof of Theorem 5.2. Combine Lemma 5.5, Lemma 5.6 and Lemma 5.10, we have the following

$$[M(4, 1)] = [(Hilb^{[3]}(\mathbb{P}^2) - \mathbb{P}^2 \times \mathbb{P}^3) \times \mathbb{P}^{11} + \mathbb{P}^2 \times (\mathbb{P}^{11} - \mathbb{P}^1) + \mathbb{P}^2 \times \mathbb{P}^{13}],$$

which leads to the theorem by direct calculation. \square

6 $M(5, 1)$ and $M(5, 2)$.

Up to isomorphism $M(5, 1)$ and $M(5, 2)$ are the only two moduli spaces with $d = 5$ such that there is no strictly semistable locus. In this section we prove the following theorem.

Theorem 6.1. $[M(5, 1)] = [M(5, 2)] = \sum_{i=0}^{26} b_{2i} \mathbb{L}^i$, such that

$$\begin{aligned}
b_0 &= b_{52} = 1; & b_2 &= b_{50} = 2; & b_4 &= b_{48} = 6; \\
b_6 &= b_{46} = 13; & b_8 &= b_{44} = 26; & b_{10} &= b_{42} = 45; \\
b_{12} &= b_{40} = 68; & b_{14} &= b_{38} = 87; & b_{16} &= b_{36} = 100; \\
b_{18} &= b_{34} = 107; & b_{20} &= b_{32} = 111; & b_{22} &= b_{30} = 112; \\
b_{24} &= b_{26} = b_{28} = 113.
\end{aligned}$$

In particular the Euler number of both moduli spaces is 1695.

◇ **Computation for $[M(5, 1)]$**

We stratify the moduli space by the form of E in (E, f) similar to what we did in the previous section. According to Lemma 3.4 we have three strata defined as follows.

$$\begin{aligned} M_1 &:= \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 4}\}; \\ M_2 &:= \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}; \\ M_3 &:= \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}\}. \end{aligned}$$

Lemma 6.2. *A pair (E, f) with $\text{rank}(E) = 5$ and $\deg(E) = -4$ is stable if and only if for any two direct summands D', D'' of D such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. See Appendix A. □

For a pair $(E, f) \in M_3$, we ask f to be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_1 & 0 & 0 & 0 & b_1 \\ a_2 & 0 & 0 & 0 & b_2 \end{pmatrix}, \quad (6.1)$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(4))$. The injectivity of f implies that $\det(f) = b_2 a_1 - b_1 a_2 \neq 0$. Moreover by Lemma 6.2 we can see that (E, f) is stable if and only if $kb_1 \neq k'b_2$ for any $(k, k') \in \mathbb{C}^2 - \{0\}$.

Lemma 6.3. $[M_3] = [\mathbb{P}^2 \times \mathbb{P}^{19}]$.

Proof. By analogous argument to Lemma 5.5 we see that M_3 is isomorphic to a projective bundle over $\text{Hilb}^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_1(5))) \simeq \mathbb{P}^{19}$. Hence the lemma follows as a direct consequence. □

We stratify M_2 into two strata as follows.

$$\begin{aligned} M_2^s &:= \{[(E, f)] \in M_2 | f|_{\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)} \text{ is surjective onto } \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\}; \\ M_2^c &:= M_2 - M_2^s. \end{aligned}$$

For a pair $(E, f) \in M_2^s$, we ask f to be represented by the following matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ b_1 & b_2 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 \end{pmatrix}, \quad (6.2)$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. The injectivity of f implies that $\det(f) = b_1 a_2 - b_2 a_1 \neq 0$. Moreover by Lemma 6.2 we can see that (E, f) is stable if and only if $kb_1 \neq k'b_2$ for any $(k, k') \in \mathbb{C}^2 - \{0\}$.

Lemma 6.4. $[M_2^s] = [Gr(2, 6) \times \mathbb{P}^{16} - \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2]$ with $Gr(2, 6)$ the Grassmannian parametrizing 2-dimensional linear subspaces of \mathbb{C}^6 .

Proof. We have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{(b_1, b_2)} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} & \xrightarrow{f_r} & R_f \longrightarrow 0. \\ & & & & \uparrow (a_1, a_2) & \nearrow \omega_f := f_r \circ (a_1, a_2) & \\ & & & & \mathcal{O}_{\mathbb{P}^2}(-3) & & \end{array} \quad (6.3)$$

Since $kb_1 \neq k'b_2, \forall (k, k') \in \mathbb{C}^2 - \{0\}$, the isomorphism classes of R_f 1-1 correspond to points in $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(2))) = Gr(2, 6)$. Denote by \mathcal{G} the tautological bundle on $Gr(2, 6)$. Then on $Gr(2, 6) \times \mathbb{P}^2$ we have the following exact sequence.

$$0 \rightarrow q^* \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow p^* \mathcal{G}^\vee \rightarrow \mathcal{R} \rightarrow 0, \quad (6.4)$$

with p and q the projections to $Gr(2, 6)$ and \mathbb{P}^2 respectively. \mathcal{R} restricted to the fiber over $[(b_1, b_2)] \in Gr(2, 6)$ is R_f . Hence isomorphism classes of (R_f, ω_f) are parametrized by the projective bundle $\mathbb{P}(p_*(\mathcal{R} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(3)))$ over $Gr(2, 6)$ with fibers isomorphic to \mathbb{P}^{16} .

However $M_2^s \subsetneq \mathbb{P}(p_*(\mathcal{R} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(3)))$ and the complement of M_2^s in $\mathbb{P}(p_*(\mathcal{R} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(3)))$ consists of all (R_f, ω_f) such that images of ω_f are contained in the torsion of R_f .

If b_1 is prime to b_2 , then R_f is torsion free. If b_1 is not prime to b_2 , then we have R_f lies in the following exact sequence.

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow R_f \rightarrow I_1(1) \rightarrow 0, \quad (6.5)$$

with H a hyperplane in \mathbb{P}^2 . The closed subset $|H| \times Hilb^{[1]}(\mathbb{P}^2) \hookrightarrow Gr(2, 6)$ parametrizes all the R_f that are not torsion free.

We write down the following diagram.

$$\begin{array}{ccc}
\mathbb{P}^2 & \xleftarrow{q} & |H| \times \text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2 \\
& & \downarrow p_1 \quad \searrow p \\
\mathcal{C}_1 & \hookrightarrow & |H| \times \mathbb{P}^2 \quad \quad \quad |H| \times \text{Hilb}^{[1]}(\mathbb{P}^2)
\end{array}$$

Those (R_f, ω_f) not in M_2^s are parametrized by the projective bundle $\mathbb{P}(p_*(p_1^*\mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$ over $|H| \times \text{Hilb}^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(\mathcal{O}_H(2))) \simeq \mathbb{P}^2$.

Hence we have $M_2^s \simeq \mathbb{P}(p_*(\mathcal{R} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3))) - \mathbb{P}(p_*(p_1^*\mathcal{O}_{\mathcal{C}_1} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(2)))$, hence the lemma. \square

For a pair $(E, f) \in M_2^c$, we ask f to be represented by the following matrix

$$\begin{pmatrix} b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_1 & a_2 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ e_1 & e_2 & 0 & a_3 & 0 \end{pmatrix}, \quad (6.6)$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and $e_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. $\det(f) \neq 0$. Lemma 6.2 implies that (E, f) is stable if and only if $kb_1 \neq k'b_2, \forall (k, k') \in \mathbb{C}^2 - \{0\}$ and $k''a_3 \neq b \cdot b_3, \forall (k'', b) \in \mathbb{C} \times H^0(\mathcal{O}_{\mathbb{P}^2}(1)) - \{(0, 0)\}$.

Lemma 6.5. $[M_2^c] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \text{Hilb}^{[2]}(\mathbb{P}^2) \times \mathbb{P}^{17} - \text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times \mathbb{P}^1]$.

Proof. We first write down the following two exact sequences.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(b_1, b_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{f_r} R_f \longrightarrow 0 \quad (6.7)$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(a_3, b_3)} \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{f_s} S_f \longrightarrow 0 \quad (6.8)$$

Because of the stability condition, we see that both R_f and S_f are torsion free and hence $R_f \simeq I_1(1)$ and $S_f \simeq I_2(2)$. On the other hand, any $I_1(1)$ ($I_2(2)$) can be put in sequence (6.7) ((6.8)).

We write down a bigger commutative diagram as follows.

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{(a_3, b_3) \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}} & \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f_s \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)}} & S_f(-1) \longrightarrow 0 \\
& & \downarrow (b_1, b_2) \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)} & & \downarrow (b_1, b_2) \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)} \oplus (b_1, b_2) & & \downarrow id_{S_f} \otimes (b_1, b_2) \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \xrightarrow{(a_3, b_3)^{\oplus 2}} & (\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})^{\oplus 2} & \xrightarrow{f_s^{\oplus 2}} & S_f^{\oplus 2} \longrightarrow 0 \\
& & \downarrow f_r \otimes id_{\mathcal{O}_{\mathbb{P}^2}(-1)} & & \downarrow f_r \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)} \oplus f_r & & \downarrow id_{S_f} \otimes f_r \\
0 & \longrightarrow & R_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{id_{R_f} \otimes (a_3, b_3)} & R_f \oplus R_f(1) & \xrightarrow{id_{R_f} \otimes f_s} & R_f \otimes S_f \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array} \tag{6.9}$$

We have another commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{(e_1, a_1) \oplus (e_2, a_2)} & (\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})^{\oplus 2} \xrightarrow{f_s^{\oplus 2}} S_f^{\oplus 2} \\
& \searrow f_r \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)} \oplus f_r & \downarrow id_{S_f} \otimes f_r \\
& & R_f \oplus R_f(1) \xrightarrow{id_{R_f} \otimes f_s} R_f \otimes S_f
\end{array} \tag{6.10}$$

We see that the isomorphism classes of $(E, f) \in M_2^c$ are parametrized by (R_f, S_f, ω_f) with $\omega_f : \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow R_f \otimes S_f$ the composed map in (6.10). We write down the following diagram.

$$\begin{array}{ccccc}
\mathbb{P}^2 & \xleftarrow{q} & Hilb^{[1]}(\mathbb{P}^2) \times Hilb^{[2]}(\mathbb{P}^2) \times \mathbb{P}^2 & & \\
& \swarrow p_1 & \downarrow p & \searrow p_2 & \\
Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2 & & Hilb^{[1]}(\mathbb{P}^2) \times Hilb^{[2]}(\mathbb{P}^2) & & Hilb^{[2]}(\mathbb{P}^2) \times \mathbb{P}^2
\end{array}$$

Denote by \mathcal{I}_1 (\mathcal{I}_2) the universal family of ideal sheaves on $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ ($Hilb^{[2]}(\mathbb{P}^2) \times \mathbb{P}^2$). Isomorphism classes of (R_f, S_f, ω_f) are parametrized by the projective bundle $\mathbb{P}(p_*(p_1^* \mathcal{I}_1 \otimes p_2^* \mathcal{I}_2 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(5)))$ over $Hilb^{[1]}(\mathbb{P}^2) \times Hilb^{[2]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_1 \otimes I_2 \otimes \mathcal{O}_{\mathbb{P}^2}(5))) \simeq \mathbb{P}^{17}$.

There are still some points in $\mathbb{P}(p_*(p_1^* \mathcal{I}_1 \otimes p_2^* \mathcal{I}_2 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(5)))$ that we must exclude. They are points (R_f, S_f, ω_f) such that the images of ω_f are contained in the torsion of $R_f \otimes S_f$.

We write down the following exact sequence.

$$0 \rightarrow I_1 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_x \rightarrow 0, \tag{6.11}$$

with \mathcal{O}_x the skyscraper sheaf supported at a single point x . Tensor (6.11) by I_2 , we get

$$0 \rightarrow \text{Tor}^1(\mathcal{O}_x, I_2) \rightarrow I_1 \otimes I_2 \rightarrow I_2 \rightarrow I_2 \otimes \mathcal{O}_x \rightarrow 0.$$

We see that the torsion of $I_1 \otimes I_2$ is isomorphic to $\text{Tor}^1(\mathcal{O}_x, I_2)$. Tensor (6.8) by \mathcal{O}_x and we get

$$0 \longrightarrow \text{Tor}^1(\mathcal{O}_x, I_2(2)) \longrightarrow \mathcal{O}_x \xrightarrow{(a_3, b_3)} \mathcal{O}_x^{\oplus 2} \longrightarrow I_2(2) \otimes \mathcal{O}_x \longrightarrow 0.$$

Hence we see that the torsion of $R_f \otimes S_f$ is either zero or isomorphic to \mathcal{O}_x with $R_f \simeq I_{\{x\}}(1)$ and $\text{Tor}^1(\mathcal{O}_x, I_2) \neq 0 \Leftrightarrow (a_3, b_3)|_x = 0$. We then construct the moduli space parametrizing (I_1, I_2) such that $I_1 \otimes I_2$ contain torsion. We first write the following diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \xleftarrow{q} & \mathbb{P}^2 \times \text{Hilb}^{[1]}(\mathbb{P}^2) \times |H| \\ & \nwarrow q_1 & \downarrow p_3 \\ \mathcal{C}_1 \hookrightarrow \mathbb{P}^2 \times |H| & & \text{Hilb}^{[1]}(\mathbb{P}^2) \times |H| \hookrightarrow \mathbb{P}(\mathcal{V}^1), \end{array} \quad (6.12)$$

where \mathcal{V}^1 is the rank 2 vector bundle on $\text{Hilb}^{[1]}(\mathbb{P}^2)$ defined as $s_*(\mathcal{I}_1 \otimes t^* \mathcal{O}_{\mathbb{P}^2}(1))$ with s and t the projection from $\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ to $\text{Hilb}^{[1]}(\mathbb{P}^2)$ and \mathbb{P}^2 respectively. We define the scheme \mathcal{Z} by the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathbb{P}(p_{3*}(q_1^*(\mathcal{O}_{\mathcal{C}_1}) \otimes q^* \mathcal{O}_{\mathbb{P}^2}(1))) \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{V}^1) & \longrightarrow & \text{Hilb}^{[1]}(\mathbb{P}^2) \times |H|. \end{array} \quad (6.13)$$

We see that $\mathbb{P}(\mathcal{V}^1)$ parametrizes the pair $([x], [C]) \in \text{Hilb}^{[1]}(\mathbb{P}^2) \times |H|$ such that $x \in C$, and \mathcal{Z} parametrizes $([x_1], [(C, x_2)]) \in \text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathcal{C}_1$ such that $x_1 \in C$. We define $\iota : \mathcal{Z} \rightarrow \text{Hilb}^{[1]}(\mathbb{P}^2) \times \text{Hilb}^{[2]}(\mathbb{P}^2)$ such that $\iota([x_1], [(C, x_2)]) = ([x_1], [x_1, x_2, C])$. It is easy to see that ι is an embedding and its image is exactly the set of points (I_1, I_2) such that $I_1 \otimes I_2$ have torsion.

$p_*(p_1^* \mathcal{I}_1 \otimes p_1^* \mathcal{I}_2 \otimes p^* \mathcal{O}_{\mathcal{Z}}) \simeq p_*(p_1^* \mathcal{I}_1 \otimes p_1^* \mathcal{I}_2) \otimes \mathcal{O}_{\mathcal{Z}}$ by the flatness of p . $p_1^* \mathcal{I}_1 \otimes p_1^* \mathcal{I}_2 \otimes p^* \mathcal{O}_{\mathcal{Z}}$ contains $p_1^* \mathcal{O}_{\mathcal{Z}_1}$ as its torsion where \mathcal{Z}_1 is the universal subscheme in $\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$. Hence we can embed \mathcal{Z} into $\mathbb{P}(p_*(p_1^* \mathcal{I}_1 \otimes p_2^* \mathcal{I}_2 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(5)))$ by taking the non-zero constant section of $p_1^* \mathcal{O}_{\mathcal{Z}_1}$.

Hence we have $M_2^c \simeq \mathbb{P}(p_*(p_1^* \mathcal{I}_1 \otimes p_2^* \mathcal{I}_2 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(5))) - \mathcal{Z}$. The lemma follows since $[\mathcal{Z}] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times \mathbb{P}^1]$. \square

Finally let $(E, f) \in M_1 - W^5$. Rewrite diagram (4.1) for $d = 5$ as the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \xrightarrow{f_{B^t}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \xrightarrow{f_q} & Q_f \longrightarrow 0. \\
& & & & \uparrow f_{A^t} & \nearrow \sigma_f = f_q \circ f_{A^t} & \\
& & & & \mathcal{O}_{\mathbb{P}^2}(-2) & &
\end{array} \quad (6.14)$$

Notice that the torsion T_f of Q_f contains neither \mathcal{O}_H nor $\mathcal{O}_{2H}(x)$ as a subsheaf with x a single point on the curve, otherwise we will have a diagram similar to diagram (5.3) which contradicts Condition 4.4. Also $H^0(T_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$. Hence we see that if $c_1(T_f) = 2H$, then $\chi(T_f) = 1$ and $Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. Moreover since T_f does not contain $\mathcal{O}_H(n)$ with $n \geq 0$, T_f is stable and hence by Theorem 5.1 there is only one stable sheaf for each curve in $|2H|$. Hence $T_f \simeq \mathcal{O}_{2H}$.

We have the following commutative diagram.

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \xrightarrow{i} & K & \longrightarrow & T_f \longrightarrow 0 \\
& & \downarrow \simeq & & \downarrow j & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \xrightarrow{f_{B^t}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \xrightarrow{f_q} & Q_f \longrightarrow 0 \\
& & & & \downarrow f_{tq} & & \downarrow \\
& & & & Q_f^{tf} & \xrightarrow{\simeq} & Q_f^{tf} \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array} \quad (6.15)$$

We stratify $M_1 - W^5$ into three strata by the form of T_f as follows.

$$\begin{aligned}
\Pi_1 &:= \{[(E, f)] \in M_1 - W^5 \mid T_f \simeq \mathcal{O}_{2H}, Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(1)\}; \\
\Pi_2 &:= \{[(E, f)] \in M_1 - W^5 \mid T_f \simeq \mathcal{O}_H(-1), Q_f^{tf} \simeq I_2(2)\}; \\
\Pi_3 &:= \{[(E, f)] \in M_1 - W^5 \mid T_f \simeq \mathcal{O}_H(-2), Q_f^{tf} \simeq I_1(2)\}.
\end{aligned}$$

A priori there is the fourth possibility that $T_f \simeq \mathcal{O}_H(-3), Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$, we will explain why this case is excluded later in the computation for $[\Pi_3]$.

Lemma 6.6. $[\Pi_1] = [\mathbb{P}^5 \times \mathbb{P}^{14} - \mathbb{P}^5 \times \mathbb{P}^4]$.

Proof. Notice that $\text{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{2H}) = 0$ for all $i \neq 1$ and $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{2H}) \simeq \mathbb{C}$, and the proof is analogous to that of Lemma 5.10. \square

Let $(E, f) \in \Pi_2$. Since $Q_f^{tf} \simeq I_2(2)$, we have the following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(a,b)} \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \xrightarrow{g} I_2(2) \rightarrow 0, \quad (6.16)$$

with $b \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and b prime to a . Take $\text{Hom}(-, \mathcal{O}_H(-1))$ on sequence (6.16) and we get

$$0 \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1)) \rightarrow \text{Ext}^1(I_2(2), \mathcal{O}_H(-1)) \xrightarrow{\tilde{g}} \text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_H(-1)) \rightarrow 0, \quad (6.17)$$

Lemma 6.7. Q_f with $T_f \simeq \mathcal{O}_H(-1)$ and $Q_f^{tf} \simeq I_2(2)$ lies in diagram (6.15) if and only if the image of following exact sequence via \tilde{g} is not zero

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow Q_f \rightarrow I_2(2) \rightarrow 0, \quad (6.18)$$

i.e. sequence (6.18) is not contained in the image of $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1))$.

Proof. The map \tilde{g} gives the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & (6.19) \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\simeq} & \mathcal{O}_{\mathbb{P}^2}(-1) & & \\ & & \downarrow & & \downarrow (a,b) & & \\ 0 \longrightarrow & \mathcal{O}_H(-1) & \longrightarrow & \widetilde{Q}_f & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} & \longrightarrow 0 \quad (*) \\ & \downarrow \simeq & & \downarrow \delta & & \downarrow g & \\ 0 \longrightarrow & \mathcal{O}_H(-1) & \longrightarrow & Q_f & \longrightarrow & I_2(2) & \longrightarrow 0 \quad (**) \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

where sequence $(*)$ is the image of sequence $(**)$ via \tilde{g} and \widetilde{Q}_f is the Cartesian product of Q_f and $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ over $I_2(2)$.

From diagram (6.19) we see that $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \widetilde{Q}_f) \simeq \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$ and $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, Q_f) \simeq \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, I_2(2))$. Moreover the map f_{tq} in diagram (6.15) factors through a surjective map $s : \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ since $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}) \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}, I_2(2))$, and s lifts to a map $\tilde{s} : \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \rightarrow \widetilde{Q}_f$ such that $f_q = \delta \circ \tilde{s}$.

If sequence $(*)$ in diagram (6.19) splits, then $\widetilde{Q}_f \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_H(-1)$ and $\delta \circ \tilde{s}$ can not be surjective. Hence $(*)$ does not split.

On the other hand, $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_H(-1)) \simeq \mathbb{C}$, hence \widetilde{Q}_f is unique up to isomorphism if $(*)$ does not split. We see that in this case $\widetilde{Q}_f \simeq \mathcal{O}_{\mathbb{P}^2} \oplus Q_f^1$

with Q_f^1 lying in the following non-splitting sequence

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow Q_f^1 \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0.$$

By Lemma 5.7 we have Q_f^1 also lies in the following sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{f_{q_1}} Q_f^1 \rightarrow 0.$$

We then have the following commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & (6.20) \\
& & \downarrow & & \downarrow & & \\
& & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \xrightarrow{\simeq} & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & & \\
& & \downarrow & & \downarrow (a,b) & & \\
0 \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \xrightarrow{f_q} & Q_f & \longrightarrow 0 \\
& \downarrow & & \downarrow f_{q_1} \oplus id_{\mathcal{O}_{\mathbb{P}^2}} & & \downarrow \simeq & \\
0 \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & Q_f^1 \oplus \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & Q_f & \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

Hence Q_f lies in diagram (6.19) and hence the lemma. \square

Lemma 6.8. $[\Pi_2] = [\text{Hilb}^{[2]}(\mathbb{P}^2) \times |H| \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^{14} - \mathbb{P}^1)]$.

Proof. Lemma 6.7 implies that for fixed $\mathcal{O}_H(-1)$ and $I_2(2)$, isomorphism classes of Q_f are parametrized by $\mathbb{P}(\text{Ext}^1(I_2(2), \mathcal{O}_H(-1))) - \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_H(-1)))$. Hence isomorphism classes of Q_f are parametrized by the following scheme

$$\mathbb{P}(\mathcal{E}xt_p^1(\mathcal{I}_2 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathcal{C}_1} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(-1))) - \text{Hilb}^{[2]}(\mathbb{P}^2) \times |H|;$$

where p and q are projections from $\mathbb{P}^2 \times \text{Hilb}^{[2]}(\mathbb{P}^2) \times |H|$ to $\text{Hilb}^{[2]}(\mathbb{P}^2) \times |H|$ and \mathbb{P}^2 respectively, \mathcal{I}_2 and \mathcal{C}_1 are the pull back of the universal ideal sheaf and the universal curve to $\mathbb{P}^2 \times \text{Hilb}^{[2]}(\mathbb{P}^2) \times |H|$ from $\mathbb{P}^2 \times \text{Hilb}^{[2]}(\mathbb{P}^2)$ and $\mathbb{P}^2 \times |H|$ respectively. Notice that we embed $\text{Hilb}^{[2]}(\mathbb{P}^2) \times |H|$ into $\mathbb{P}(\mathcal{E}xt_p^1(\mathcal{I}_2 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathcal{C}_1} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(-1)))$ by taking the nonzero constant section of the line bundle $\mathcal{H}om_p(q^* \mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathcal{C}_1} \otimes q^* \mathcal{O}_{\mathbb{P}^2}(-1)) \simeq p_* \mathcal{O}_{\mathcal{C}_1} \simeq \mathcal{O}_{\text{Hilb}^{[2]}(\mathbb{P}^2) \times |H|}$.

Analogously the space parametrizing (Q_f, σ_f) is the difference of two projective bundles with fibers isomorphic to $\mathbb{P}(H^0(Q_f(2))) \simeq \mathbb{P}^{14}$ and $\mathbb{P}(H^0(\mathcal{O}_H(1))) \simeq \mathbb{P}^1$ respectively over the space parametrizing Q_f . Hence the lemma. \square

Now we do the computation for $[\Pi_3]$ and we will also explain why the case that $T_f \simeq \mathcal{O}_H(-3)$, $Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ is not possible. We see that the map

f_{tq} in diagram (6.15) is not surjective on global sections. We first write down the following commutative diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K & \xrightarrow{j} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \xrightarrow{f_{tq}} & Q_f^{tf} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \simeq \\
 0 & \longrightarrow & G & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4+n} & \xrightarrow{g} & Q_f^{tf} \longrightarrow 0 \\
 & & \downarrow \tau & & \downarrow \tilde{\tau} & & \\
 & & \mathcal{O}_{\mathbb{P}^2}^{\oplus n} & \xrightarrow{\simeq} & \mathcal{O}_{\mathbb{P}^2}^{\oplus n} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array} \tag{6.21}$$

where $n = 1$ if $Q_f^{tf} \simeq I_1(2)$ and $n = 2$ if $Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$.

From diagram (6.21) we see that $H^i(G(1-i)) = 0$ for $i > 0$, hence by Castelnuovo-Mumford regularity $G(1)$ is globally generated. Therefore the map $\tau \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)} : G(1) \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus n}$ must be surjective on global sections, since any sheaf generated by a proper subspace of $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ is the ideal sheaf of a curve or a point in \mathbb{P}^2 . Hence $h^0(K(1)) = h^0(G(1)) - nh^0(\mathcal{O}_{\mathbb{P}^2}(1))$. So if $n = 2$, $Q_f^{tf} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$, then we have $h^0(K(1)) = 2$ which implies that K can not contain $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$ as a subsheaf. Hence we only have $n = 1$ and $Q_f^{tf} \simeq I_1(2)$.

$T_f \simeq \mathcal{O}_H(-2)$ hence $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), T_f) = 0$, the inclusion ι in diagram (6.15) is unique up to isomorphisms of $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$ for fixed K . Hence f_{B^t} is determined by inclusion j and hence is determined by f_{tq} . To parametrize f_{B^t} is equivalent to parametrizing the surjective map f_{tq} , hence equivalent to parametrize $\tilde{\tau}$. We first assume $Q_f^{tf} \simeq I_{[0,0,1]}(2)$, then g can be represented by $\text{diag}(x_0^2, x_0x_1, x_0x_2, x_1x_2, x_1^2)$. $\tilde{\tau}$ can be represented by a 5×1 matrix $\underline{h} := (h_0, h_1, h_2, h_3, h_4)^t$ with $h_i \in \mathbb{C}$. We want to parametrize \underline{h} .

We see that G can be generated by 6 generators $\langle \epsilon_0, \epsilon_1, \epsilon_2, \eta_0, \eta_1, \eta_2 \rangle$ in $H^0(G(1))$ with two syzygies $(x_0\epsilon_0 + x_1\epsilon_1 - x_2\epsilon_2 = 0, x_0\eta_0 + x_1\eta_1 - x_2\eta_2 = 0)$.

The map τ is determined by $\tau^0 := h^0(\tau \otimes id_{\mathcal{O}_{\mathbb{P}^2}(1)}) : H^0(G(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, also τ is induced by $\tilde{\tau}$. Hence τ^0 is determined by \underline{h} and we can write down explicitly the images of ϵ_i and η_i as follows

$$\begin{aligned}
 \tau^0(\epsilon_0) &= h_2x_1 - h_1x_2; & \tau^0(\epsilon_1) &= h_0x_2 - h_2x_0; & \tau^0(\epsilon_2) &= h_0x_1 - h_1x_0 \\
 \tau^0(\eta_0) &= h_3x_1 - h_4x_2; & \tau^0(\eta_1) &= h_1x_2 - h_3x_0; & \tau^0(\eta_2) &= h_1x_1 - h_4x_0.
 \end{aligned}$$

We have diagram (6.21) if and only if τ^0 is surjective. In other words, the following 3×6 matrix has rank 3.

$$Mat_\tau := \begin{pmatrix} 0 & -h_2 & -h_1 & 0 & -h_3 & -h_4 \\ h_2 & 0 & h_0 & h_3 & 0 & h_1 \\ -h_1 & h_0 & 0 & -h_4 & h_1 & 0 \end{pmatrix}$$

By direct computation we see that $rank(Mat_\tau) < 3 \Leftrightarrow h_1h_2 - h_0h_3 = h_1^2 - h_0h_4 = h_1h_3 - h_2h_1 = 0$. Hence we know that f_{tq} are parametrized by $P_\tau := \mathbb{P}^4 - \{h_1h_2 - h_0h_3 = h_1^2 - h_0h_4 = h_1h_3 - h_2h_1 = 0\}$.

One can easily compute that $[P_\tau] = [\mathbb{P}^4 - (\mathbb{P}^1 + \mathbb{A}^2 + \mathbb{A}^1)]$. Moreover we can cover $Hilb^{[1]}(\mathbb{P}^2)$ by finitely many Zariski open subsets U_i such that $(I_1(2), \tilde{\tau})$ with $[I_1] \in U_i$ are parametrized by $U_i \times P_\tau$. For example, we can take U_i such that $p_*(\mathcal{I}_1 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(2))|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus 5}$, where p and q are the projections from $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ to $Hilb^{[1]}(\mathbb{P}^2)$ and \mathbb{P}^2 respectively and \mathcal{I}_1 the universal ideal sheaf.

$(I_1(2), \tilde{\tau})$ determines Q_f and analogously we know that (Q_f, σ_f) are parametrized by a difference of two projective bundles over the space parametrizing Q_f . Hence we have the following lemma as a direct consequence.

Lemma 6.9. $[\Pi_3] = [Hilb^{[1]}(\mathbb{P}^2) \times (\mathbb{P}^4 - (\mathbb{P}^1 + \mathbb{A}^2 + \mathbb{A}^1)) \times (\mathbb{P}^{14} - 1)]$.

We have already known that $[W^5] = \mathbb{P}^{14} \times [Hilb^{[6]}(\mathbb{P}^2) - \Omega_2^{[6]}]$. We have the following lemma, the proof of which is postponed to the appendix.

Lemma 6.10. $[\Omega_2^{[6]}] = \mathbb{L}^{11} + 3\mathbb{L}^{10} + 8\mathbb{L}^9 + 18\mathbb{L}^8 + 30\mathbb{L}^7 + 39\mathbb{L}^6 + 38\mathbb{L}^5 + 28\mathbb{L}^4 + 15\mathbb{L}^3 + 6\mathbb{L}^2 + 2\mathbb{L}^1 + 1$.

Proof. See Appendix B. □

Proof of Theorem 6.1 for $M(5, 1)$. We have

$$[M(5, 1)] = [M_3] + [M_2^s] + [M_2^c] + \sum_{i=1}^3 [\Pi_i] + ([Hilb^{[6]}(\mathbb{P}^2)] - [\Omega_2^{[6]}]) \times [\mathbb{P}^{14}].$$

Combine Lemma 6.3, Lemma 6.4, Lemma 6.5, Lemma 6.6, Lemma 6.8, Lemma 6.9 and Lemma 6.10, we get the result by direct computation. □

◇ **Computation for $[M(5, 2)]$**

We stratify the moduli space by the form of E in (E, f) and we have three strata defined as follows.

$$\begin{aligned} M_2 &:= \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}\}; \\ M_3 &:= \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}; \\ M'_3 &:= \{[(E, f)] \in M(5, 1) | E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\}. \end{aligned}$$

Here we use notation M'_3 instead of M_4 because we want to specify the lower index of the subspace to be $h^0(F)$ with F any sheaf in it.

Lemma 6.11. *A pair (E, f) with $\text{rank}(E) = 5$ and $\deg(E) = -3$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. See Appendix A. □

For a pair $(E, f) \in M'_3$, we ask f to be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ d & 0 & 0 & b & 0 \\ c & 0 & 0 & a & 0 \end{pmatrix}, \quad (6.22)$$

where $b \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, $a \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$, $c \in H^0(\mathcal{O}_{\mathbb{P}^2}(4))$ and $d \in H^0(\mathcal{O}_{\mathbb{P}^2}(3))$. The injectivity of f implies that $\det(f) = ad - bc \neq 0$. Moreover by Lemma 6.2 we can see that (E, f) is stable if and only if b is prime to a .

Lemma 6.12. $[M'_3] = [\text{Hilb}^{[2]}(\mathbb{P}^2) \times \mathbb{P}^{18}]$.

Proof. We have the following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(a,b)} \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \rightarrow I_2(2) \rightarrow 0, \quad (6.23)$$

and for every $[I_2] \in \text{Hilb}^{[2]}(\mathbb{P}^2)$, $I_2(2)$ lies in sequence (6.23). Hence analogously to Lemma 5.5, M'_3 is isomorphic to a projective bundle over $\text{Hilb}^{[2]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(I_2(5))) \simeq \mathbb{P}^{18}$. Hence the lemma. □

For a pair $(E, f) \in M_3$, we ask f to be represented by the following matrix

$$\begin{pmatrix} B & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ A & 0 & 0 \end{pmatrix},$$

where A is a 1×3 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(3))$ and B a 2×3 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. We have parametrized isomorphism classes of B in the previous section when we computed $[M(4, 1)]$, and the class of the parametrizing space of B is $[Hilb^{[3]}(\mathbb{P}^2) - |H| \times \mathbb{P}^3 + |H|]$ by Lemma 5.6 and Lemma 5.9. Hence we have the following lemma.

Lemma 6.13. $[M_3] = [(Hilb^{[3]}(\mathbb{P}^2) - |H| \times \mathbb{P}^3 + |H|) \times \mathbb{P}^{17} - |H| \times \mathbb{P}^2]$.

Proof. M_3 is the union of a projective bundle over $Hilb^{[3]}(\mathbb{P}^2) - \Omega_1^{[3]}$ with fiber isomorphic to $\mathbb{P}(H^0(I_3(5))) \simeq \mathbb{P}^{17}$ and a difference of two projective bundles over $|H|$ with fibers isomorphic to \mathbb{P}^{17} and $\mathbb{P}(H^0(\mathcal{O}_H(2))) \simeq \mathbb{P}^2$ respectively. Hence the lemma. \square

We stratify M_2 into two strata as follows.

$$M_2^s := \{[(E, f)] \in M_2 | f_{rs} : \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow{f|_{\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}}} E \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \text{ is injective}\};$$

$$M_2^c := M_2 - M_2^s.$$

For a pair $(E, f) \in M_2^s$, we ask f to be represented by the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A & 0 & 0 & B \end{pmatrix},$$

where A is a 3×2 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and B a 3×1 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$.

We stratify M_2^s by the form of B into two strata as follows.

$$\Xi_1 := \{[(E, f)] \in M_2^s | B \simeq (x_0, x_1, x_2)^t\};$$

$$\Xi_2 := M_2^s - \Xi_1.$$

We see that if $B \simeq (x_0, x_1, x_2)^t$ then (E, f) always satisfies the stability condition. We have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{(x_0, x_1, x_2)} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \xrightarrow{f_0} & E_0 \longrightarrow 0, \\ & & & & \uparrow f_{A^t} & \nearrow \xi_f := f_0 \circ f_{A^t} & \\ & & & & \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} & & \end{array} \quad (6.24)$$

with E_0 rank 2 bundle which is the dual of the kernel of the surjective map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{(x_0, x_1, x_2)^t} \mathcal{O}_{\mathbb{P}^2}(1)$. Isomorphism classes of ξ_f are parametrized by $Gr(2, 15)$

since $h^0(E_0(2)) = 15$. Moreover $\det(f) \neq 0 \Leftrightarrow$ the image of ξ_f is a rank two subsheaf of $E_0 \Leftrightarrow \text{Im}(\xi_f)$ is not contained in a rank one subsheaf of E_0 .

Assume $\text{Im}(\xi_f)$ is contained in a rank one subsheaf $E_1 \subsetneq E_0$. Since E_0 is locally free, we ask E_1 to be a line bundle. Hence either $E_1 \simeq \mathcal{O}_{\mathbb{P}^2}$ or $E_1 \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. Notice that for any n a map $\mathcal{O}_{\mathbb{P}^2}(n) \rightarrow E_0$ always factors through map f_0 in diagram (6.24).

Since $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0) \simeq \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$, all inclusions $i : \mathcal{O}_{\mathbb{P}^2} \hookrightarrow E_0$ are parametrized by $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) - \{0\}$. Moreover $\forall i, i' \in H^0(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) - \{0\}, i \neq i', \text{Im}(i) \cap \text{Im}(i') = \emptyset$. Hence all ξ_f such that $\text{Im}(\xi_f)$ is contained in $\mathcal{O}_{\mathbb{P}^2} \simeq E_1 \subsetneq E_0$ are parametrized by $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(2))) \times \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3})) \simeq Gr(2, 6) \times \mathbb{P}^2$.

Let $\iota \in \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}) - \{0\}$. All inclusions $j : \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow E_0$ that do not factor through $\iota : \mathcal{O}_{\mathbb{P}^2}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^2}$ are parametrized by $\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0) - \tilde{i}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}))$, where $\tilde{i} : \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0) \hookrightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0)$ is the map induced by ι . Moreover $\forall j, j' \in H^0(\mathcal{O}_{\mathbb{P}^2}(-1), E_0) - \{0\}, j \neq j', \text{Im}(j) \cap \text{Im}(j') = \emptyset$, and $\forall \iota, \iota' \in H^0(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}) - \{0\}, \iota \neq \iota', \text{Im}(\iota) \cap \text{Im}(\iota') = \emptyset$. Hence all ξ_f such that $\text{Im}(\xi_f)$ are contained in $\mathcal{O}_{\mathbb{P}^2}(-1)$ but not $\mathcal{O}_{\mathbb{P}^2}$ in E_0 are parametrized by $Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times (\mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), E_0)) - \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}, E_0)) \times \mathbb{P}(\text{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}))) \simeq \mathbb{P}^2 \times (\mathbb{P}^7 - \mathbb{P}^2 \times \mathbb{P}^2)$.

We have the following lemma as a direct consequence.

Lemma 6.14. $[\Xi_1] = [Gr(2, 15) - Gr(2, 5) \times \mathbb{P}^2 - \mathbb{P}^2 \times (\mathbb{P}^7 - \mathbb{P}^2 \times \mathbb{P}^2)]$.

For a pair $(E, f) \in \Xi_2$, we ask f to be represented by the following matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & b_1 \\ a_5 & a_6 & 0 & 0 & b_2 \end{pmatrix}, \quad (6.25)$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $a_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$. By Lemma 6.2 (E, f) is stable if and only if $kb_1 \neq k'b_2, ka_1 \neq k'a_2, \forall (k, k') \in \mathbb{C} - \{0\}$.

Matrix (6.25) is similar to (6.6) and also our procedure for $[\Xi_2]$ is analogous to that for $[M_2^c]$ with $M_2^c \subset M(5, 1)$. We write down the following two exact sequences.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(b_1, b_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{f_r} R_f \longrightarrow 0 \quad (6.26)$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{(a_1, a_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{f_s} S_f \longrightarrow 0 \quad (6.27)$$

$R_f \simeq I_1(1)$ and either $S_f \simeq I_4(2)$ or S_f lies in the following exact sequence.

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow S_f \rightarrow I_1(1) \rightarrow 0. \quad (6.28)$$

Isomorphism classes of (R_f, S_f) are parametrized by $\text{Hilb}^{[1]}(\mathbb{P}^2) \times \text{Gr}(2, 6)$.

We have two commutative diagrams as follows.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-3) & \xrightarrow{(a_1, a_2) \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)}} & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} & \xrightarrow{f_s \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)}} & S_f(-1) \longrightarrow 0 \\ & & \downarrow (b_1, b_2) \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-2)} & & \downarrow (b_1, b_2)^{\oplus 2} & & \downarrow \text{id}_{S_f} \otimes (b_1, b_2) \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} & \xrightarrow{(a_1, a_2)^{\oplus 2}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} & \xrightarrow{f_s^{\oplus 2}} & S_f^{\oplus 2} \longrightarrow 0 \\ & & \downarrow f_r \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-2)} & & \downarrow f_r^{\oplus 2} & & \downarrow \text{id}_{S_f} \otimes f_r \\ 0 & \longrightarrow & R_f \otimes \mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{\text{id}_{R_f} \otimes (a_1, a_2)} & R_f^{\oplus 2} & \xrightarrow{\text{id}_{R_f} \otimes f_s} & R_f \otimes S_f \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \quad (6.29)$$

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{(a_3, a_4) \oplus (a_5, a_6)} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \xrightarrow{f_s^{\oplus 2}} S_f^{\oplus 2} \\ & & \downarrow f_r^{\oplus 2} \quad \downarrow \text{id}_{S_f} \otimes f_r \\ & & R_f^{\oplus 2} \xrightarrow{\text{id}_{R_f} \otimes f_s} R_f \otimes S_f. \end{array} \quad (6.30)$$

We see that the isomorphism classes of $(E, f) \in \Xi_2$ are parametrized by (R_f, S_f, ω_f) with $\omega_f : \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow R_f \otimes S_f$ the composed map in (6.30). Hence firstly we have a projective bundle over $\text{Hilb}^{[1]}(\mathbb{P}^2) \times \text{Gr}(2, 6)$ with fibers isomorphic to $\mathbb{P}(H^0(R_f \otimes S_f(2))) \simeq \mathbb{P}^{15}$, which contains Ξ_2 as an open subset. The complement of Ξ_2 in that projective bundle is the set of all (R_f, S_f, ω_f) such that $\text{Im}(\omega_f)$ are contained in the torsion of $R_f \otimes S_f$.

Torsion free S_f are parametrized by $\text{Gr}(2, 6) - \mathbb{P}^2 \times \mathbb{P}^2$. For S_f torsion free, $R_f \otimes S_f$ has torsion if and only if $(a_1, a_2)|_x = 0$ with $R_f \simeq I_x(1)$, and if there is nonzero torsion, it is isomorphic to \mathcal{O}_x . Define $\mathcal{V}_1^i := p_*(\mathcal{I}_1 \otimes q^* \mathcal{O}_{\mathbb{P}^2}(i))$ with \mathcal{I}_1 , p, q the same as before. \mathcal{V}_2^1 and \mathcal{V}_1^2 are two vector bundles of rank 2 and 5 respectively over $\text{Hilb}^{[1]}(\mathbb{P}^2)$. Hence (R_f, S_f, ω_f) with S_f torsion free and $\text{Im}(\omega_f)$ contained in the torsion of $R_f \otimes S_f$ are parametrized by $\text{Gr}(2, \mathcal{V}_1^2) -$

$Gr(2, \mathcal{V}_1^1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1))) \cup Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times \mathbb{P}(\mathcal{V}_1^1)$, where $Gr(2, \mathcal{V}_1^i)$ is the relative Grassmannian of vector bundle \mathcal{V}_1^i . We can see that

$$\begin{aligned} & [Gr(2, \mathcal{V}_1^1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1))) \cup Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times \mathbb{P}(\mathcal{V}_1^1)] \\ = & [Gr(2, \mathcal{V}_1^1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))] + [Gr(2, h^0(\mathcal{O}_{\mathbb{P}^2}(1))) \times \mathbb{P}(\mathcal{V}_1^1)] - [\mathbb{P}(\mathcal{V}_1^1)] \\ = & [Hilb^{[1]}(\mathbb{P}^2) \times (\mathbb{P}^2 + \mathbb{P}^2 \times \mathbb{P}^1 - \mathbb{P}^1)]. \end{aligned}$$

Now let S_f lie in sequence (6.28). Write $R_f \simeq I_x(1)$ and $I_y(1)$ the quotient of S_f in (6.28). If $x \neq y$, then $R_f \otimes I_y(1)$ is torsion free and in this case (R_f, S_f, ω_f) with $\text{Im}(\omega_f)$ contained in the torsions of $R_f \otimes S_f$ are parametrized by a projective bundle over $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2 \times \mathbb{P}^2 - Gr(2, \mathcal{V}_1^1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))$ with fibers isomorphic to $\mathbb{P}(H^0(I_1(1) \otimes \mathcal{O}_H(1))) \simeq \mathbb{P}^2$.

Finally we have a projective bundle over $Gr(2, \mathcal{V}_1^1) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(1)))$ with fibers isomorphic to $\mathbb{P}(H^0(\mathcal{O}_x) \oplus H^0(I_1(1) \otimes \mathcal{O}_H(1))) \simeq \mathbb{P}^3$ parametrizing (R_f, S_f, ω_f) such that $x = y$ and $\text{Im}(\omega_f)$ are contained in the torsions of $R_f \otimes S_f$. Hence we have the following lemma.

Lemma 6.15. $[\Xi_2] = [Hilb^{[1]}(\mathbb{P}^2) \times ((Gr(2, 6) \times \mathbb{P}^{15} - Gr(2, 5) - \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 - \mathbb{P}^2 \times \mathbb{P}^3 - \mathbb{P}^1 + \mathbb{P}^2 \times \mathbb{P}^2 + \mathbb{P}^1 \times \mathbb{P}^2 + \mathbb{P}^2))]$.

For a pair $(E, f) \in M_2^c$, we ask f to be represented by the following matrix

$$\begin{pmatrix} b_1 & b_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A_1 & A_2 & 0 & B \end{pmatrix},$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$, A_i is a 3×1 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ and B a 3×2 matrix with entries in $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. (E, f) is stable $\Leftrightarrow kb_1 \neq k'b_2, \forall (k, k') \in \mathbb{C}^2 - \{0\}$ and $[B] \in M_B$, where M_B is the parametrizing space of B we have constructed in the previous section when we computed $[M(4, 1)]$. Notice that $[M_B] = [Hilb^{[3]}(\mathbb{P}^2) - \mathbb{P}^2 \times \mathbb{P}^3 + |H|]$ by Lemma 5.6 and Lemma 5.9. We write down the following two exact sequences.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \xrightarrow{f_{B^t}} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{f_r} R_f \longrightarrow 0 \quad (6.31)$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{(b_1, b_2)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{f_s} S_f \longrightarrow 0 \quad (6.32)$$

$S_f \simeq I_1(1)$ and either $R_f \simeq I_3(2)$ or R_f lies in the following exact sequence.

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow R_f \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0. \quad (6.33)$$

Isomorphism classes of (R_f, S_f) are parametrized by $M_B \times \text{Hilb}^{[1]}(\mathbb{P}^2)$.

We have two commutative diagrams as follows.

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} & \xrightarrow{((b_1, b_2) \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)})^{\oplus 2}} & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 4} & \xrightarrow{(f_s \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)})^{\oplus 2}} & S_f(-1)^{\oplus 2} \longrightarrow 0 \\
& \downarrow f_{B^t} \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)} & & \downarrow f_{B^t}^{\oplus 2} & & \downarrow \text{id}_{S_f} \otimes (b_1, b_2) & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} & \xrightarrow{(b_1, b_2)^{\oplus 3}} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 6} & \xrightarrow{f_s^{\oplus 3}} & S_f^{\oplus 3} \longrightarrow 0 \\
& \downarrow f_r \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-1)} & & \downarrow f_r^{\oplus 2} & & \downarrow \text{id}_{S_f} \otimes f_r & \\
0 & \longrightarrow & R_f \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\text{id}_{R_f} \otimes (b_1, b_2)} & R_f^{\oplus 2} & \xrightarrow{\text{id}_{R_f} \otimes f_s} & R_f \otimes S_f \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array} \tag{6.34}$$

$$\begin{array}{ccccc}
\mathcal{O}_{\mathbb{P}^2}(-2) & \xrightarrow{A_1^t \oplus A_2^t} & \mathcal{O}_{\mathbb{P}^2}^{\oplus 6} & \xrightarrow{f_s^{\oplus 2}} & S_f^{\oplus 3} \\
& & \downarrow f_r^{\oplus 2} & & \downarrow \text{id}_{S_f} \otimes f_r \\
& & R_f^{\oplus 2} & \xrightarrow{\text{id}_{R_f} \otimes f_s} & R_f \otimes S_f.
\end{array} \tag{6.35}$$

We see that the isomorphism classes of $(E, f) \in M_2^c$ are parametrized by (R_f, S_f, ω_f) with $\omega_f : \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow R_f \otimes S_f$ the composed map in (6.35). Hence firstly we have a projective bundle over $M_B \times \text{Hilb}^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(R_f \otimes S_f(2))) \simeq \mathbb{P}^{16}$, which contains M_2^c as an open subset.

We need to exclude all the points (R_f, S_f, ω_f) that $\text{Im}(\omega_f)$ are contained in the torsion of $R_f \otimes S_f$. Firstly let R_f lie in sequence (6.33), then the torsion of $R_f \otimes S_f$ is isomorphic to $\mathcal{O}_H(-1) \otimes I_1(1)$. Those (R_f, S_f, ω_f) are parametrized by a projective bundle over $(\cup_{i=0}^2 \mathbb{A}^i) \times \text{Hilb}^{[1]}(\mathbb{P}^2)$ with fibers isomorphic to $\mathbb{P}(H^0(\mathcal{O}_H(1) \otimes I_1(1))) \simeq \mathbb{P}^2$.

Let $R_f \simeq I_3(2)$. Denote $S_f \simeq I_x(1)$. The torsion of $R_f \otimes S_f$ is a linear subspace of $\mathcal{O}_x^{\oplus 2} \simeq \mathbb{C}^2$ which is the kernel of $B^t|_x$. If $x \notin \text{Supp}(\mathcal{O}_{\mathbb{P}^2}(2)/R_f)$, $R_f \otimes S_f$ is torsion free. If $\text{Supp}(\mathcal{O}_{\mathbb{P}^2}(2)/R_f) = \{x, y, z\}$ with $y, z \neq x$, for simplicity we let $x = [0, 0, 1]$, $y = [0, 1, 0]$ and $z = [1, 0, 0]$ and we can ask matrix B to have the following form.

$$\begin{pmatrix} x_1 & 0 \\ x_0 & x_0 \\ 0 & x_2 \end{pmatrix}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$.

Let $R_f \simeq I_{\{x, 2y\}}$, then we can take B as follows.

$$\begin{pmatrix} x_0 & 0 \\ x_2 & x_0 \\ 0 & x_1 \end{pmatrix}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$.

Let $R_f \simeq I_{\{2x, y\}}$, then we can take B as follows.

$$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_2 \end{pmatrix}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$.

Let $R_f \simeq I_{\{3x\}}$, then we can take B as follows.

$$\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ kx_2 & x_1 \end{pmatrix}, \text{ for any } k \in \mathbb{C}.$$

Hence for this case $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x$ if $k \neq 0$, $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x^{\oplus 2}$ if $k = 0$.

We see that the projective bundle $\mathbb{P}(\mathcal{V}_1^1)$ as defined before on $Hilb^{[1]}(\mathbb{P}^2)$ parametrizes all (x, C) with x a single point and C a curve of degree 1 passing through x . Hence we have the universal family $\overline{\mathcal{C}}_1 \subset \mathbb{P}^2 \times \mathbb{P}(\mathcal{V}_1^1)$. Denote \mathcal{Z}_1 to be the universal family of subschemes in $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^2$ and $\pi : \mathbb{P}(\mathcal{V}_1^1) \rightarrow Hilb^{[1]}(\mathbb{P}^2)$ the projection. Define $\overline{\mathcal{C}}_1^0 := \overline{\mathcal{C}}_1 - (\pi \times id_{\mathbb{P}^2})^* \mathcal{Z}_1$. Denote $\mathbb{P}(\mathcal{V}_1^1)^{[2]}$ the relative Hilbert scheme of 2-points on $\mathbb{P}(\mathcal{V}_1^1)$ over $Hilb^{[1]}(\mathbb{P}^2)$. There is a natural embedding $\iota : \mathbb{P}(\mathcal{V}_1^1) \hookrightarrow \mathbb{P}(\mathcal{V}_1^1)^{[2]}$ sending every point to the double-point supported at it. We have the following diagram.

$$\begin{array}{ccc} \overline{\mathcal{C}}_1^0 \times_{\mathbb{P}(\mathcal{V}_1^1) \times_{\pi} \mathbb{P}(\mathcal{V}_1^1)} \overline{\mathcal{C}}_1^0 - p^* \Delta(\mathbb{P}(\mathcal{V}_1^1)) & \xrightarrow{\delta'} & \mathcal{X} \\ p \downarrow & & \downarrow p' \\ \mathbb{P}(\mathcal{V}_1^1) \times_{\pi} \mathbb{P}(\mathcal{V}_1^1) - \Delta(\mathbb{P}(\mathcal{V}_1^1)) & \xrightarrow{\delta} & \mathbb{P}(\mathcal{V}_1^1)^{[2]} - \iota(\mathbb{P}(\mathcal{V}_1^1)), \end{array} \quad (6.36)$$

with Δ the diagonal embedding and \mathcal{X} is defined to make diagram (6.36) a Cartesian diagram. Notice that a priori \mathcal{X} may not exist, but if it exists, it parametrizes isomorphism classes of (R_f, S_f, ω_f) with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{x, y, z\}}(2) \in N_4^0$ and $\text{Im}(\omega) \subset Tor(R_f \otimes S_f)$.

Lemma 6.16. \mathcal{X} exists and $[\mathcal{X}] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times (\mathbb{P}^2 - \mathbb{P}^1) \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)]$.

Proof. Take an affine cover of $\text{Hilb}^{[1]}(\mathbb{P}^2) = \cup_i U_i$ with $U_i \simeq \mathbb{A}^2$. It is enough to prove the lemma with $\text{Hilb}^{[1]}(\mathbb{P}^2)$ replaced by U_i . Write $\mathcal{Z}_1|_{U_i}$, $\mathbb{P}(\mathcal{V}_1^1)|_{U_i}$, $\mathbb{P}(\mathcal{V}_1^1)^{[2]}|_{U_i}$, $\overline{\mathcal{C}}_1|_{U_i}$ and $\overline{\mathcal{C}}_1^0|_{U_i}$ the pull back of these schemes via the open embedding $U_i \hookrightarrow \text{Hilb}^{[1]}(\mathbb{P}^2)$. Then we have that $\mathcal{Z}_1|_{U_i} \simeq U_i$, $\mathbb{P}(\mathcal{V}_1^1)|_{U_i} \simeq U_i \times \mathbb{P}^1$, $\mathbb{P}(\mathcal{V}_1^1)^{[2]}|_{U_i} \simeq U_i \times \mathbb{P}^2$, $\overline{\mathcal{C}}_1|_{U_i} \simeq U_i \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\overline{\mathcal{C}}_1^0|_{U_i} \simeq U_i \times \mathbb{P}^1 \times \mathbb{A}^1$. Hence diagram (6.36) becomes the following commutative diagram.

$$\begin{array}{ccc} U_i \times (\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)) \times \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{\delta'_i} & \mathcal{X}_i \\ \downarrow p_i & & \downarrow p'_i \\ U_i \times (\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)) & \xrightarrow{\delta_i} & U_i \times (\mathbb{P}^2 - \iota(\mathbb{P}^1)), \end{array} \quad (6.37)$$

with $\mathcal{X}_i \simeq U_i \times (\mathbb{P}^2 - \iota(\mathbb{P}^1)) \times \mathbb{A}^1 \times \mathbb{A}^1$. Hence the lemma. \square

Isomorphism classes of (R_f, S_f) with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{2x, y\}}(2) \in N_5^0$ and $\text{Im}(\omega) \subset \text{Tor}(R_f \otimes S_f)$ are parametrized by $\mathbb{P}(\mathcal{V}_1^1) \times_{\text{Hilb}^{[1]}(\mathbb{P}^2)} \overline{\mathcal{C}}_1^0 - \Delta$, where Δ is defined by the following Cartesian diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\quad} & \overline{\mathcal{C}}_1^0 \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{V}_1^1) & \xrightarrow{id} \mathbb{P}(\mathcal{V}_1^1) & \xrightarrow{\pi} \text{Hilb}^{[1]}(\mathbb{P}^2) \end{array}$$

Lemma 6.17. $[\mathbb{P}(\mathcal{V}_1^1) \times_{\text{Hilb}^{[1]}(\mathbb{P}^2)} \overline{\mathcal{C}}_1^0 - \Delta] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)]$.

Proof. Replace $\text{Hilb}^{[1]}(\mathbb{P}^2)$ by U_i as in the proof of Lemma 6.16 and we see the lemma immediately. \square

The normal sheaf of $\overline{\mathcal{C}}_1$ in $\mathbb{P}^2 \times \mathbb{P}(\mathcal{V}_1^1)$ is locally free over $\overline{\mathcal{C}}_1^0$. We denote by $\mathcal{N}_{\mathcal{C}}^0$ the total space of the normal bundle over $\overline{\mathcal{C}}_1^0$. Then isomorphism classes of (R_f, S_f, ω_f) with $S_f \simeq I_x(1)$, $R_f \simeq I_{\{x, 2y\}}(2) \in N_4^0$ and $\text{Im}(\omega) \subset \text{Tor}(R_f \otimes S_f)$ are parametrized by $\mathcal{N}_{\mathcal{C}}^0$ and we have the following lemma.

Lemma 6.18. $[\mathcal{N}_{\mathcal{C}}^0] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) \times (\mathbb{P}^1 - 1)]$.

Proof. $[\mathcal{N}_{\mathcal{C}}^0] = [\mathbb{A}^1 \times (\overline{\mathcal{C}}_1 - (\pi \times id_{\mathbb{P}^2})^* \mathcal{Z}_1)]$. $[(\pi \times id_{\mathbb{P}^2})^* \mathcal{Z}_1] = [\mathbb{P}^1 \times \text{Hilb}^{[1]}(\mathbb{P}^2)]$ and $[\overline{\mathcal{C}}_1] = [\text{Hilb}^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 \times \mathbb{P}^1]$. Hence the lemma. \square

If $R_f \simeq I_{\{3x\}} \in N_4^0$ and $Tor(R_f \otimes I_x) \simeq \mathcal{O}_x$, then R_f , viewed as an ideal of $\widehat{\mathcal{O}}_{\mathbb{P}^2, x} \simeq \mathbb{C}[[x_0, x_1]]$, is generated by $(kx_0 - x_1^2, \mathfrak{m}^3)$ with \mathfrak{m} the maximal ideal in $\mathbb{C}[[x_0, x_1]]$ and $k \in \mathbb{C}^*$. Hence such R_f are parametrized by (x_0, k) for a given $x \in \mathbb{P}^2$. Hence isomorphism classes of these (R_f, S_f, ω_f) such that $\text{Im}(\omega) \subset Tor(R_f \otimes S_f)$ are parametrized by $\mathbb{P}(\mathcal{V}_1^1) \times (\mathbb{A}^1 - \{0\})$.

Finally, let $S_f \simeq I_x(1)$, $R_f \simeq I_{\{3x\}}(2) \in N_4^0$ and $Tor(R_f \otimes S_f) \simeq \mathcal{O}_x^{\oplus 2}$, then R_f is determined by x since $R_f(-2) \simeq I_x^2$. Hence isomorphism classes of these (R_f, S_f, ω_f) such that $\text{Im}(\omega) \subset Tor(R_f \otimes S_f)$ are parametrized by $Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}(H^0(\mathcal{O}_x^{\oplus 2})) \simeq Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1$.

Lemma 6.19. $[M_2^c] = [M_B \times Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^{16} - \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 - \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1 - 1) - \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2 - \mathbb{P}^1)]$, with $[M_B] = [Hilb^{[3]}(\mathbb{P}^2) - \mathbb{P}^2 \times \mathbb{P}^3 + |H|]$.

Proof. $[M_2^c] = [M_B \times Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^{16} - \mathbb{P}^2 \times |H| \times Hilb^{[1]}(\mathbb{P}^2) - \mathbb{P}(\mathcal{V}_1^1) \times (\mathbb{A}^1 - 1) - Hilb^{[1]}(\mathbb{P}^2) \times \mathbb{P}^1 - \mathcal{X} - \mathbb{P}(\mathcal{V}_1^1) \times_{Hilb^{[1]}(\mathbb{P}^2)} \overline{\mathcal{C}}_1^0 + \Delta - \mathcal{N}_{\mathcal{C}}^0]$.

By Lemma 6.16, Lemma 6.17 and Lemma 6.18, we get the lemma by direct computation. \square

Proof of Theorem 6.1 for $M(5, 2)$. We have

$$[M(5, 2)] = [M'_3] + [M_3] + [\Xi_1] + [\Xi_2] + [M_2^c].$$

Combine Lemma 6.12, Lemma 6.13, Lemma 6.14, Lemma 6.15 and Lemma 6.19, we get the result by direct computation. \square

Appendix

A Proofs of Lemma 6.2 and Lemma 6.11.

Lemma A.1 (Lemma 6.2). *A pair (E, f) with $\text{rank}(E) = 5$ and $\text{deg}(E) = -4$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. We first prove the lemma for $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. We want to show that if $\exists E' \subset E$, E' is a direct sum of line bundles with $\mu(E') > \mu(E)$ and $f^{-1}(E') \simeq E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$, then $\exists D, D' \subset E$ as two direct summands with $D \simeq D'$ and $\mu(D) > \mu(E)$, such that $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D'$. With no loss

of generality, we assume that E' has the form $\bigoplus_i \mathcal{O}_{\mathbb{P}^2}(n_i)^{\oplus a_i}$ with $a_i > 0$ and $n_i - n_{i+1} = 1$.

Let $E' \simeq E'' \subset E$ with E'' not a direct summand of E . Then E'' has to be one of the following three cases:

- (1) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;
- (2) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$;
- (3) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

By Nakayama's lemma, we know that $E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ can't be the preimage of any direct summand of E and also $f^{-1}(E'') \simeq E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \Rightarrow f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D$ with D the smallest direct summand of E containing E'' .

So we assume that $f^{-1}(E'') = E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with E' a direct summand of E isomorphic to E'' .

Let E'' be in case (1). By the assumption we have $f(E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$. On the other hand, write $E = \mathcal{O}_{\mathbb{P}^2} \oplus E' \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, so for the other direct summand $\mathcal{O}_{\mathbb{P}^2}$ we have $f(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. Hence $f((\mathcal{O}_{\mathbb{P}^2} \oplus E') \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, and hence we get $D = E' \oplus \mathcal{O}_{\mathbb{P}^2}$ and $D' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ which has to be a direct summand of E .

Case (2) is analogous to case (1).

Let E'' be in case (3). By the assumption we have $f(E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Write $E = E' \oplus L$ with $L \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. We can ask f to identify $L \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with the summand $\mathcal{O}_{\mathbb{P}^2}(-2)$ in E .

Denote $f_o : E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow E''$ to be the restriction of f . If $f_o((\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, then we have $D = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \subset E'$ and $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D$.

If $f_o((\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \not\subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, f_o induces an isomorphism from the direct summand $\mathcal{O}_{\mathbb{P}^2}(-1) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ of $E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ to the direct summand $\mathcal{O}_{\mathbb{P}^2}(-2)$ of E'' . Hence we can ask f_o to identify these two direct summands. Write $E' = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus L'$ with $L' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$, then we have $f_o(L' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. Moreover because f identifies $L \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ with the summand $\mathcal{O}_{\mathbb{P}^2}(-2)$ in E , $f((L \oplus L') \otimes \mathcal{O}_{\mathbb{P}^2}(-1))$ is contained in the direct summand $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ of E , hence we have $D = L \oplus L'$ and D' is the direct summand $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ containing $f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1))$.

This finishes the proof for $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

Let $E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}$. We have the following six possibilities for E'' .

- (4) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}$;
- (5) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$;
- (6) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;
- (7) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;
- (8) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;
- (9) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

Analogously we see that $E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ can not be the preimage of any direct summand of E and also $f^{-1}(E'') \simeq E'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \Rightarrow f(D \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D$ with D the smallest direct summand of E containing E'' . Besides let E_3'' and E_4'' be the bundles in case (6) and (7) respectively. $f^{-1}(E_3'') = E_4'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \Rightarrow f((\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, and $f^{-1}(E_4'') = E_3'' \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \Rightarrow f(\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1)$.

Hence we then assume $E' \simeq E''$ is a direct summand of E and $f^{-1}(E'') = E' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$.

For case (4), by assumption we have $f(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ hence $f((\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$.

The bundles in case (5), case (8) and case (9) can not be direct summands of E , hence these three cases are done.

For case (6), by assumption we have $f((\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ hence $f((\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.

For case (7), by assumption we have $f(\mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ hence $f((\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.

This finishes the proof for the whole lemma. \square

Lemma A.2 (Lemma 6.11). *A pair (E, f) with $\text{rank}(E) = 5$ and $\text{deg}(E) = -3$ is stable if and only if for any two direct summands D', D'' of E such that $D' \simeq D''$ and $f(D' \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \subset D''$, we have $\mu(D') < \mu(E)$.*

Proof. We use the same notations as in the proof of Lemma A.1, we list out all the possibilities of E'' as follows.

Let $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$.

(1) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

Let $E \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

(2) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(3) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(4) $E'' \subset \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$;

Let $E \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

(5) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}$;

(6) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$;

(7) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(8) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(9) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$;

(10) $E'' \subset \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ and $E'' \simeq \mathcal{O}_{\mathbb{P}^2}(1) \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$.

The proofs for cases (1) (5) (6) (7) (8) (9) are the same as those for cases (1) (4) (5) (6) (7) (8) in Lemma A.1 respectively. Case (10) is analogous to case (3) in Lemma A.1. Cases (2) (3) (4) are analogous to case (1). Hence the lemma. \square

B Proof of Lemma 6.10.

Lemma B.1 (Lemma 6.10). $[\Omega_2^{[6]}] = \mathbb{L}^{11} + 3\mathbb{L}^{10} + 8\mathbb{L}^9 + 18\mathbb{L}^8 + 30\mathbb{L}^7 + 39\mathbb{L}^6 + 38\mathbb{L}^5 + 28\mathbb{L}^4 + 15\mathbb{L}^3 + 6\mathbb{L}^2 + 2\mathbb{L}^1 + 1$.

Proof. Denote by \mathcal{C}_2 the universal curve in $\mathbb{P}^2 \times |2H|$ and $\mathcal{C}_2^{[6]}$ the relative Hilbert scheme of 6-points on \mathcal{C}_2 . Then we have a natural surjective map $\xi : \mathcal{C}_2^{[6]} \rightarrow \Omega_2^{[6]}$. The fiber of ξ over $[I_{\{x_1, \dots, x_6\}}]$ is the space parametrizing all curves passing through x_1, \dots, x_6 and hence isomorphic to $\mathbb{P}(H^0(I_{\{x_1, \dots, x_6\}}(2))) \simeq \mathbb{P}^n$ with $n \leq 2$. We stratify $\Omega_2^{[6]}$ by n into three strata, i.e. $\Omega_2^{[6]} = \coprod_{n=0}^2 \mathcal{S}_n$. Define $\mathcal{R}_n := \xi^{-1}(\mathcal{S}_n)$, then $\mathcal{R}_n \simeq \mathbb{P}(p_*(\mathcal{I}(2)|_{\mathbb{P}^2 \times \mathcal{S}_n}))$ is a projective bundle over \mathcal{S}_n with fibers isomorphic to \mathbb{P}^n .

Denote by \mathcal{C}_2^o the family of integral curves in $|2H|$. \mathcal{C}_2^o is open in \mathcal{C}_2 .

Lemma B.2. $[\mathcal{C}_2^{o[6]}] = [(|2H| - \text{Sym}^2(|H|)) \times \mathbb{P}^6]$ with $\text{Sym}^2(|H|)$ the symmetric power of order 2 of $|H|$.

Proof. The subspace $|2H|^o$ in $|2H|$ parametrizing integral curves is isomorphic to $|2H| - \text{Sym}^2(|H|)$. $\mathcal{C}_2^{o[6]}$ is a projective bundle over $|2H|^o$ with fibers isomorphic to \mathbb{P}^6 . Hence the lemma. \square

Denote $\mathcal{C}_2^I \rightarrow (\text{Sym}^2(|H|) - |H|)$ and $\mathcal{C}_2^D \rightarrow |H|$ the family of reducible curves and non-reduced curves in $|2H|$ respectively. Let C^I be a reducible curve in $|2H|$ and C^D a non-reduced curve. Denote $R_n^I (\mathcal{R}_n^I) = \mathcal{R}_n \cap \text{Hilb}^{[6]}(C^I)$ ($\mathcal{C}_2^{I[6]}$) and $R_n^D (\mathcal{R}_n^D) = \mathcal{R}_n \cap \text{Hilb}^{[6]}(C^D)$ ($\mathcal{C}_2^{D[6]}$). Then we have the following lemma

Lemma B.3. $[\mathcal{R}_n^D] = [R_n^D \times |H|]$, for $n = 0, 1, 2$.

Proof. We can take an affine cover of $|H|$, write $|H| = \cup_j V_j$ such that $\mathcal{C}_2^D|_{V_j} \simeq V_j \times C^D$. Hence the lemma. \square

Denote $\mathfrak{m} = \langle x, y \rangle$ the maximal ideal of $\mathbb{S} := \mathbb{C}[[x, y]]$. To study R_n^D for $n = 0, 1, 2$, we write down a table for ideals in $\mathbb{S}/(x^2)$ as Table I.

Table I		
Ideals I of \mathbb{S} containing (x^2)		
Co-length of I	Ideal I	$I \cap (\mathfrak{m}^2 - \mathfrak{m}^3)$
1	\mathfrak{m}	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
2	$\mathfrak{m}^2 + (kx + k'y)\mathbb{S}, (k, k') \neq 0$	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
3	\mathfrak{m}^2	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
	$\mathfrak{m}^3 + (x + ky^2)\mathbb{S}$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
4	$x^2\mathbb{S} + (ky^2 + k'xy)\mathbb{S} + \mathfrak{m}^3, (k, k') \neq 0$	$\mathbb{C}x^2 \oplus \mathbb{C}(ky^2 + k'xy)$
	$(x + ky^2 + k'y^3)\mathbb{S} + \mathfrak{m}^4, k \neq 0$	$\mathbb{C}x^2$
	$(x + k'y^3)\mathbb{S} + \mathfrak{m}^4$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
5	$x^2\mathbb{S} + \mathfrak{m}^3$	$\mathbb{C}x^2$
	$x^2\mathbb{S} + (xy + ky^2 + k'y^3)\mathbb{S} + \mathfrak{m}^4, k \neq 0$	$\mathbb{C}x^2$
	$x^2\mathbb{S} + (xy + k'y^3)\mathbb{S} + \mathfrak{m}^4$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(x + ky^3 + k'y^4)\mathbb{S} + \mathfrak{m}^5, k \neq 0$	$\mathbb{C}x^2$
	$(x + k'y^4)\mathbb{S} + \mathfrak{m}^5$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
6	$x^2\mathbb{S} + (kxy^2 + k'y^3)\mathbb{S} + \mathfrak{m}^4, (k, k') \neq 0$	$\mathbb{C}x^2$
	$x^2\mathbb{S} + (xy + ky^3 + k'y^4)\mathbb{S} + \mathfrak{m}^5, k \neq 0$	$\mathbb{C}x^2$
	$x^2\mathbb{S} + (xy + k'y^4)\mathbb{S} + \mathfrak{m}^5$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(x + ky^3 + k'y^4 + k''y^5)\mathbb{S} + \mathfrak{m}^6, (k, k'') \neq 0$	$\mathbb{C}x^2$
	$(x + k''y^5)\mathbb{S} + \mathfrak{m}^6$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$

Let C_r^D be the reduced curve supported on C^D , then $C_r^D \simeq \mathbb{P}^1$. Define S^i to be the subset in $Hilb^{[6]}(C^D)$ consisting of $[I_i^r \cap I_{6-i}]$ with $[I_i^r] \in Hilb^{[i]}(C_r^D)$ and i maximal for this expression, and $S_n^i = S^i \cap R_n^D$. We have the following lemma.

Lemma B.4. S_n^i are empty except the following 9 terms:

1. $[S_2^6] = [\mathbb{P}^6]$;
2. $[S_1^4] = [\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{A}^1]$;
3. $[S_1^3] = [\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{A}^4]$;
4. $[S_0^2] = [\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{A}^5]$, $[S_2^2] = [\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{A}^3]$;
5. $[S_0^1] = [\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{A}^4]$, $[S_2^1] = [\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{A}^3]$;
6. $[S_0^0] = [\mathbb{P}^2 \times \mathbb{P}^1 \times (\mathbb{A}^4 + 2\mathbb{A}^3 + \mathbb{A}^2) + \mathbb{P}^2 \times \mathbb{A}^4 + \mathbb{P}^2(\mathbb{P}^3 - \mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{A}^3]$,
 $[S_1^0] = [\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{A}^1]$.

We omit the proof of Lemma B.4 since it can be done by elementary analysis and computation case by case. Lemma B.4 together with Lemma B.6 gives the form of $[\mathcal{R}_n^D]$ for $n = 0, 1, 2$.

To compute $[\mathcal{R}_n^I]$, we first define $\tilde{\mathcal{C}}_2^I$ by the following Cartesian diagram.

$$\begin{array}{ccc} \tilde{\mathcal{C}}_2^I & \xrightarrow{\pi_1} & \mathcal{C}_2^I \\ \downarrow & & \downarrow \\ \mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2) & \xrightarrow{\pi} & Sym^2(\mathbb{P}^2) - \mathbb{P}^2, \end{array}$$

where π is the quotient by the free action of the order two permutation group σ_2 . The action of σ_2 lifts to $\tilde{\mathcal{C}}_2^{I[6]}$ with $\mathcal{C}_2^{I[6]}$ the quotient. Recall that $R_n^I(\mathcal{R}_n^I) = \mathcal{R}_n \cap Hilb^{[6]}(C^I) \cap \mathcal{C}_2^{I[2]}$, we then denote $\widetilde{\mathcal{R}}_n^I = \pi_2^{-1}(\mathcal{R}_n^I)$ with π_2 the lift of π . \mathcal{R}_n^I is the quotient of $\widetilde{\mathcal{R}}_n^I$ by the action of σ_2 .

Lemma B.5. $[\widetilde{\mathcal{R}}_n^I] = [(\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)) \times R_n^I]$ for $n = 0, 1, 2$.

Proof. Analogous to Lemma B.3, we can take an affine cover of $\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2)$ which trivializes $\tilde{\mathcal{C}}_2^I$. \square

Table II Ideals I of \mathbf{S} containing (xy)		
Co-length of I	Ideal I	$I \cap (\mathfrak{m}^2 - \mathfrak{m}^3)$
1	\mathfrak{m}	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
2	$\mathfrak{m}^2 + (kx + k'y)\mathbf{S}, (k, k') \neq 0$	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
3	\mathfrak{m}^2	$\mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$
	$\mathfrak{m}^3 + (x + ky^2)\mathbf{S}$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$\mathfrak{m}^3 + (y + kx^2)\mathbf{S}$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
4	$xy\mathbf{S} + (kx^2 + k'y^2)\mathbf{S} + \mathfrak{m}^3, (k, k') \neq 0$	$\mathbb{C}xy \oplus \mathbb{C}(kx^2 + k'y^2)$
	$(x + ky^3)\mathbf{S} + \mathfrak{m}^4$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(y + kx^3)\mathbf{S} + \mathfrak{m}^4$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
5	$xy\mathbf{S} + \mathfrak{m}^3$	$\mathbb{C}xy$
	$xy\mathbf{S} + (x^2 + ky^3)\mathbf{S} + \mathfrak{m}^4, k \neq 0$	$\mathbb{C}xy$
	$xy\mathbf{S} + x^2\mathbf{S} + \mathfrak{m}^4$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$xy\mathbf{S} + (y^2 + kx^3)\mathbf{S} + \mathfrak{m}^4, k \neq 0$	$\mathbb{C}xy$
	$xy\mathbf{S} + y^2\mathbf{S} + \mathfrak{m}^4$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
	$(x + ky^4)\mathbf{S} + \mathfrak{m}^5$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(y + kx^4)\mathbf{S} + \mathfrak{m}^5$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
6	$xy\mathbf{S} + (kx^3 + k'y^3)\mathbf{S} + \mathfrak{m}^4, (k, k') \neq 0$	$\mathbb{C}xy$
	$xy\mathbf{S} + (x^2 + ky^4)\mathbf{S} + \mathfrak{m}^5, k \neq 0$	$\mathbb{C}xy$
	$xy\mathbf{S} + x^2\mathbf{S} + \mathfrak{m}^5$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$xy\mathbf{S} + (y^2 + kx^4)\mathbf{S} + \mathfrak{m}^5, k \neq 0$	$\mathbb{C}xy$
	$xy\mathbf{S} + y^2\mathbf{S} + \mathfrak{m}^5$	$\mathbb{C}xy \oplus \mathbb{C}y^2$
	$(x + ky^5)\mathbf{S} + \mathfrak{m}^6$	$\mathbb{C}x^2 \oplus \mathbb{C}xy$
	$(y + kx^5)\mathbf{S} + \mathfrak{m}^6$	$\mathbb{C}xy \oplus \mathbb{C}y^2$

Denote by 0 the only singular point in C^I . $C^I - \{0\} = \mathbb{A}^1 \sqcup \mathbb{A}^1$. $\widehat{\mathcal{O}}_{C^I, 0} \simeq \mathbf{S}/(xy) = \mathbb{C}[[x, y]]/(xy)$. We make a table for ideals of $\mathbf{S}/(xy)$ as Table II.

σ_2 acts on $\text{Hilb}^{[6]}(C^I)$ by exchanging the two irreducible components of C^I . Write $\text{Hilb}^{[6]}(C^I) = H^x \sqcup H^s \sqcup H^y$ such that $\sigma_2(H^x) = H^y$ and $\sigma_2(H^s) = H^s$. We put the lower indexes to those three spaces to stand for their intersections with \mathcal{R}_n , for instance $H_n^x = \mathcal{R}_n \cap H^x$. $\sigma_2(H_n^x) = H_n^y$, $\sigma_2(H_n^s) = H_n^s$.

Lemma B.6. 1. $[H_0^x] = [\mathbb{A}^6 + 2\mathbb{A}^5 + 3\mathbb{A}^4 + 3\mathbb{A}^3 + 2\mathbb{A}^2 - 1]$;

2. $[H_1^x] = [\mathbb{A}^6 + 2\mathbb{A}^5 + 2\mathbb{A}^4 + 2\mathbb{A}^3 + 2\mathbb{A}^2 + 2\mathbb{A}^1]$;

3. $[H_2^x] = [\mathbb{P}^6]$;

4. $[H_0^s] = [\mathbb{A}^6 + \mathbb{A}^4 \times \mathbb{P}^1 + \mathbb{A}^2 \times \mathbb{P}^1 + \mathbb{P}^1]$;

5. $H_1^s = \emptyset$;

6. $H_2^s = \emptyset$.

Proof. $\forall I_6 \in \text{Hilb}^{[6]}(C^I)$, $\exists 0 \leq i \leq 6$, such that $I_6 = I_i^0 \cap I_{6-i}$ with $[I_i^0] \in \text{Hilb}^i(\{0\})$ and $[I_{6-i}] \in \text{Hilb}^{[6-i]}(C^I - \{0\}) = \text{Hilb}^{[6-i]}(\mathbb{A}^1 \sqcup \mathbb{A}^1)$. Then the lemma can be proved by elementary analysis and computation case by case. \square

Lemma B.7. $[\mathcal{R}_n^I] = [H_n^x \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))] + [H_n^s \times \text{Sym}^2(\mathbb{P}^2) - \mathbb{P}^2]$.

Proof. $\sigma_2(H_n^x \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))) = H_n^y \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))$ and $\sigma_2(H_n^s \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))) = H_n^s \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta(\mathbb{P}^2))$. Hence the lemma. \square

Now combine Lemma B.2, Lemma B.3, Lemma B.4, Lemma B.5, Lemma B.6 and Lemma B.7, we get $[\mathcal{R}_n]$ for $n = 0, 1, 2$. On the other hand $[\Omega_2^{[6]}] = \sum_{n=0}^2 [\mathcal{S}_n]$ and $[\mathcal{S}_n \times \mathbb{P}^n] = [\mathcal{R}_n]$. We then get Lemma 6.10 by direct computation. \square

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